New Geometrical Perspective of Fuzzy ART and Fuzzy ARTMAP Learning

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ABSTRACT

In this paper we introduce new useful, geometric concepts regarding categories in Fuzzy ART and Fuzzy ARTMAP, which shed more light into the process of category competition eligibility upon the presentation of input patterns. First, we reformulate the competition of committed nodes with uncommitted nodes in an F_2 layer as a commitment test very similar to the vigilance test. Next, we introduce a category's match and choice regions, which are the geometric interpretation of the vigilance and commitment test respectively. After examining properties of these regions we reach three results applicable to both Fuzzy ART and Fuzzy ARTMAP. More specifically, we show that only one out of these two tests is required; which test needs to be performed depends on the values of the vigilance parameter ρ and the choice parameter a. Also, we show that for a specific relation of ρ and a, the vigilance ρ does not influence the training or performance phase of Fuzzy ART and Fuzzy ARTMAP. Finally, we refine a previously published upper bound on the size of categories created during training in Fuzzy ART and Fuzzy ARTMAP.

Keywords: adaptive resonance theory, self-organization, Fuzzy ART, Fuzzy ARTMAP

1. INTRODUCTION

Fuzzy-ART[1] (FA) and Fuzzy-ARTMAP[2] (FAM) are two neural network architectures with roots in the *adaptive resonance theory*[3]. While FA is intended for unsupervised clustering purposes, FAM is capable of building maps between clusters of two separate spaces (known as *input* and *output spaces*) via a supervised learning scheme. While a single FA module comprises FA, FAM consists of two FA modules interconnected with each other through an inter-ART module, which is responsible for forming the appropriate cluster associations. Moreover, as a special application, FAM can be used as classifier system. In this particular case, the output space of the mapping coincides with the set of class labels corresponding to the patterns of the input space and therefore the role of the output space FA module becomes trivial. The means of learning for both networks is the summarization of similar training patterns into clusters, which we will define as *FA categories* and are used both in FA and FAM. FA categories are the building block of knowledge/memory representation for both architectures. The forming of FA categories itself is achieved in a self-organizing manner; learning in FA and FAM does not involve the minimization of any objective function. There are many desirable properties of learning and characteristics associated to FA/FAM. First, they are both capable of off-line (batch) and on-line (incremental) learning. Under fast learning rule[1,2] assumptions, both exhibit fast, stable and finite learning: the networks' knowledge stabilizes relatively fast after a finite number of *list presentations* (epochs). Under the same assumptions, the FAM classifier achieves 100% correct classification on its training set. Furthermore, they both feature outlier detection mechanisms that identify input patterns not typical of previously experienced inputs. Also, due to the specifics of their neural architecture, responses of FA and FAM to specified inputs are easily explained, in contrast to other neural network models, where in general it is difficult to explain why an input pattern **x** produced an output **y**. However, both architectures have been criticized about their lack of a noise removal mechanisms, which would have lessen the effects of noise present in the input data and minimize the risk of over-fitting. Properties of learning for FA and FAM can be found in their original references[1,2], as well as in the work of others[4,5].

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We must note that significant insight into the functionality of FA/FAM has been gained by studying certain geometrical concepts related to FA categories. For example, the fact that FA categories can be represented as hyper-rectangles in the input domain has aided in the development of several properties of learning and has increased our level of understanding about the inner workings of these architectures. In this paper we will attempt to explain what makes a FA category eligible to select an input pattern and we will present some consequences of our newly introduced geometrical concepts, which apply both to FA and FAM.

2. FUZZY-ART CATEGORY BASICS

Before proceeding with the introduction of new geometrical concepts revolving around FA categories, we believe that it is necessary to provide limited background on certain key elements. We assume that the reader is already familiar with the basic concepts about FA/FAM. We begin by introducing some useful notation. Let R be the set of real numbers and $U_M \in [0,1]^M$ denote the closure of the M-dimensional unit hyper-cube that serves as an input space for any FA-module. We define as $|\cdot|:U_{2M} \rightarrow R$ to be the L_1-norm for the U_{2M} domain and $\|\cdot\|_1:U_M \rightarrow R$ to be the L_1-norm for the U_M domain. Additionally, we define the min-operator $\wedge:U_{2M}\times U_{2M} \rightarrow U_{2M}$, such that, if $\mathbf{w}_1$, $\mathbf{w}_2 \in U_{2M}$ and $\mathbf{w}_3 = \mathbf{w}_1 \wedge \mathbf{w}_2$, then the m^{th} component w_{3m} of vector $\mathbf{w}_3$ is $w_{3m}=\min\{w_{1m}, w_{2m}\}$. In the FA/FAM literature it is customary to denote as $\mathbf{a} \in U_M$ as a pattern of the input space and as $I=[\mathbf{a}\ \mathbf{a}^c] \in U_{2M}$ to be its complemented coded version, where $\mathbf{a}^c=\mathbf{1}-\mathbf{a}$ and $\mathbf{1}$ is the all-ones vector. Vector I serves as the input vector to the FA modules. On grounds of convenience, we take the liberty of using $\mathbf{x}$ and $\mathbf{x}^e=[\mathbf{x}\ \mathbf{x}^c]=[\mathbf{x}\ \mathbf{1}-\mathbf{x}]$ for the input pattern and its complement coded form. Note, that all aforementioned quantities are row vectors. The information of each FA category j is stored in the FA module's *template*, which is a vector of the form $\mathbf{w}_j=[\mathbf{u}_j\ \mathbf{v}_j^c] \in U_{2M}$ and $\mathbf{u}_j$, $\mathbf{v}_j \in U_M$; we will call the latter vectors *template elements*. Due to FA/FAM's learning schemes, for every template it always holds that $u_{jm} \le v_{jm}$ with $m=1..M$. The *size* $s(\mathbf{w}_j)$ of a category with template $\mathbf{w}_j$ is defined as

$$s(\mathbf{w}_j) = M - |\mathbf{w}_j| = \|\mathbf{v}_j - \mathbf{u}_j\|_1 = \sum_{m=1}^{M} (v_{jm} - u_{jm}). \tag{1}$$

Note, that for every input pattern $\mathbf{x} \in U_M$ and template $\mathbf{w}_j$ it holds

$$\begin{aligned} |\mathbf{w}_j| &= M - \|\mathbf{v}_j - \mathbf{u}_j\|_1 = M - \sum_{m=1}^{M} (v_{jm} - u_{jm}) \\ |\mathbf{x}^e \wedge \mathbf{w}_j| &= 2M - \sum_{m=1}^{M} [\max\{x_m, v_{mj}\} - \min\{x_m, u_{mj}\}] \end{aligned}. \tag{2}$$

Based on Equations 1 and 2, we define as the *distance* of a pattern $\mathbf{x} \in U_M$ from a category with template $\mathbf{w}_j$ the quantity

$$dis(\mathbf{x}, \mathbf{w}_j) = |\mathbf{w}_j| - |\mathbf{x}^e \wedge \mathbf{w}_j| = \sum_{m=1}^{M} [(\max\{x_m, v_{jm}\} - v_{jm}) + (u_{jm} - \min\{x_m, u_{jm}\})] - M . \tag{3}$$

There is a one-to-one correspondence between FA categories, templates and committed nodes in the F_2 layer of each FA module. Each one of these nodes represents an FA category; the category's description is stored in its *top-down* weight vector, which constitutes its template. Uncommitted nodes in the F_2 layer of FA modules do not correspond to categories and represent the "blank" memory of the system. It has been shown[1,2,4] that FA categories can be geometrically represented as hyper-rectangles embedded in the FA module's input space U_M. An example, when $M=2$, is shown in Figure 1 depicted in the next page. The union of the shaded area in Figure 1 and the boundaries of the rectangle defined by $\mathbf{u}_j$ and $\mathbf{v}_j$, is called *representation region* of category j. Also depicted in the same figure, $dis(\mathbf{x},\mathbf{w}_j)$ reflects the minimum L_1 distance (also known as *city-block* or *Manhattan* distance) between pattern $\mathbf{x}$ and category's j representation region. Note that, if $\mathbf{x}$ were inside or on the borders of the rectangle, its distance from category j would have been 0.

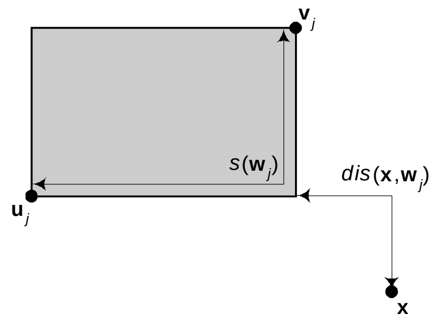

Figure 1: Geometric representation of FA category *j* assuming a 2-dimensional input space.

Learning in a FA module is achieved by creating categories and updating them as necessary in light of new information. For a FA category, incorporating new evidence in the form of a training pattern, is attained by increasing its size and simultaneously reducing the distance between the pattern and the category. The learning law that implements this idea for a category *j* being updated due to a pattern **x** is

$$\mathbf{w}_j^{new} = \gamma\left(\mathbf{x} \wedge \mathbf{w}_j^{old}\right) + (1-\gamma)\mathbf{w}_j^{old} \,. \tag{4}$$

where $\gamma \in (0,1]$ is a learning rate parameter. Due to Equations 1 through 4 we deduce that

$$s(\mathbf{w}_j^{new}) = s(\mathbf{w}_j^{new}) + \gamma \; dis(\mathbf{x}, \mathbf{w}_j^{old}) \,. \tag{5}$$

$$dis(\mathbf{x}, \mathbf{w}_j^{new}) = (1-\gamma) dis(\mathbf{x}, \mathbf{w}_j^{old}) \,. \tag{6}$$

As special case, when $\gamma=1$ (*fast learning* assumption), a category *j* that will be modified upon presentation of pattern **x** will increase its size so that its new representation region contains **x**. In such a case, we say that the updated category *j* *encodes* pattern **x**. In all other cases, where $\gamma<1$ (*slow learning* assumption), we will say that pattern **x** *updated* category *j*. Figure 2 illustrates the 2 cases in a 2-dimensional setting

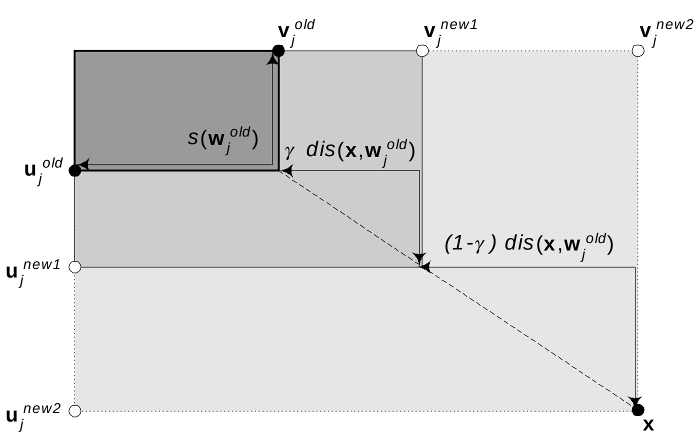

Figure 2: FA category modifications - slow vs. fast learning.

In the above figure, under slow learning pattern **x** updates $\mathbf{w}_j^{old}$ to $\mathbf{w}_j^{new1}$, while under fast learning $\mathbf{w}_j^{new2}$ expands enough to encode **x**. Notice that under both learning assumptions the new representation region includes the previous one. As a general comment, FA categories are never destroyed during learning and can only increase in size; destruction of a category would be equivalent to partial loss of the FA module's knowledge.

Except of the learning rate γ, FA modules have 3 more parameters: the *vigilance parameter* $\rho\in[0,1]$, the *choice parameter* $a>0$ and the order of pattern presentation. The latter one can be thought of as a parameter affecting only the module's training phase (for P training patterns we have $P!$ distinct orders), since the order, in which training patterns are presented to the module, affects the characteristics and the number of the module's categories. The vigilance and choice parameters are the only ones that affect both the training and the performance phase of a module. Two important quantities related to FA categories are the *category match function* $\rho(\mathbf{w}|\mathbf{x})$ (CMF) and the *category choice function* $T(\mathbf{w}|\mathbf{x})$ (CCF - also known as *bottom-up input* or *activation function*), which are defined below:

$$\rho(\mathbf{w}\,|\,\mathbf{x}) = \frac{\left|\mathbf{x}^e \wedge \mathbf{w}\right|}{M}. \tag{7}$$

$$T(\mathbf{w}\,|\,\mathbf{x}) = \frac{\left|\mathbf{x}^e \wedge \mathbf{w}\right|}{\left|\mathbf{w}\right| + a}. \tag{8}$$

Based on Equations 1 and 3, the CMF and CCF can be alternatively expressed via geometry-based quantities as

$$\rho(\mathbf{w}\,|\,\mathbf{x}) = \frac{M - s(\mathbf{w}) - dis(\mathbf{x},\mathbf{w})}{M}. \tag{9}$$

$$T(\mathbf{w}\,|\,\mathbf{x}) = \frac{M - s(\mathbf{w}) - dis(\mathbf{x},\mathbf{w})}{M - s(\mathbf{w}) + a}. \tag{10}$$

The above functions play a central role in the two modes of operation of a FA module (training and performance). The CMF value of a category with respect to an input pattern is the quantity used in the comparison to the FA module's vigilance parameter ρ. On the other hand, the CCF value of a category with respect to a pattern $\mathbf{x}$ is used to determine the winning node in the F_2 layer of FA modules during node competition for $\mathbf{x}$; the node with the highest CCF value is rendered to be the winner.

3. THE COMMITMENT TEST

The comparison of CMF values to the vigilance parameter ρ constitutes the *vigilance test* (VT), which acts as a screening device for categories before the node competition takes place. If a given input pattern during either training or performance phase does not fit the characteristics of a category j, then j will fail the VT and is disqualified from the node competition. Therefore, the VT can be regarded as a novelty detection mechanism that is able of pointing out non-typical patterns with respect to existing categories in the FA module. The VT is expressed as

$$\rho(\mathbf{w}_j\,|\,\mathbf{x}) \geq \rho. \tag{11}$$

Categories fail the test, when their CMF value is less than ρ. However, the VT is not the only component of novelty detection. During node competition both committed and uncommitted nodes participate. Assuming that uncommitted nodes have a template of $\mathbf{w}_u=\mathbf{1}$, they feature constant CMF and CCF values of

$$\rho(\mathbf{w}_u\,|\,\mathbf{x}) = \rho_u = 1. \tag{12}$$

$$T(\mathbf{w}_u\,|\,\mathbf{x}) = T_u = \frac{M}{2M + a}. \tag{13}$$

From Equations 11 and 12 it is apparent that uncommitted nodes always pass the VT, since $\rho\in[0,1]$. In order for a category j to have a chance of winning the competition and be the one that best explains the presence of a given pattern, it must definitely have a CCF value higher or equal than the one of uncommitted nodes, that is,

$$T(\mathbf{w}_j \mid \mathbf{x}) \geq T_u \,. \tag{14}$$

For the reason we mentioned above we can view the above comparison of CCF values as a test similar to the VT, which determines the eligibility of a category j to compete with other existing categories in the same FA module for a given input pattern. Furthermore, it acts as an outlier detector too.

Definition 1

We define as *commitment test* (CT) of a category j featuring a template $\mathbf{w}_j$ with respect to an input pattern $\mathbf{x}$ the comparison of its CCF value, $T(\mathbf{w}_j|\mathbf{x})$, to the CCF value of an uncommitted node, T_u. We say that category j passes the CT, when $T(\mathbf{w}_j|\mathbf{x}) \geq T_u$.

CT is a useful conceptual device, as it will become apparent later in this paper. Taking into account the facts presented so far, we can formulate the following definition:

Definition 2

A category j that passes both the VT and CT, when presented with an input pattern $\mathbf{x}$ during either training or performance phase, will be called *eligible* to select $\mathbf{x}$. The set of all existing categories that are eligible to select $\mathbf{x}$ is called the *candidate set* $S(\mathbf{x})$.

4. CATEGORY REGIONS

At this point we are ready to define various *FA category regions*. Unfortunately, proofs of the properties presented in this and the following section are left out due to lack of space. However, the reader might be able to verify these properties in the 2-dimensional case with the aid of figures that are provided. The first region, a category's representation region, has been already introduced in the previous section; we will just provide its formal definition based on Equations 2 and 3.

Definition 3

We define as *representation region* $R_j=R(\mathbf{w}_j)$ of a category j with template $\mathbf{w}_j$ the following subset of U_M

$$R(\mathbf{w}_j) = \{\mathbf{x} \in U_M \mid \mathbf{x} \wedge \mathbf{w}_j = \mathbf{w}_j\} \quad \Leftrightarrow \quad R(\mathbf{w}_j) = \{\mathbf{x} \in U_M \mid dis(\mathbf{x}, \mathbf{w}_j) = 0\}. \tag{15}$$

It has also been shown pictorially in Figure 2 that, if $\mathbf{x} \notin R(\mathbf{w}_j)$, then $R(\mathbf{w}_j) \subset R(\mathbf{x}^e \wedge \mathbf{w}_j)$, otherwise $R(\mathbf{w})=R(\mathbf{x}^e \wedge \mathbf{w}_j)$. This implies that $R(\mathbf{w}) \subseteq R(\mathbf{w} \wedge \mathbf{x}^e)$ for any $\mathbf{x} \in U_M$ and, when a category is being modified, its representation region expands (includes more points of the input space). Next, we will proceed with a definition that adorns the VT with a new geometrical interpretation.

Definition 4

We define as *match (vigilance) region* $V_j=V(\mathbf{w}_j|\rho)$ of a category j with template $\mathbf{w}_j$ for a particular value ρ of the vigilance parameter the following subset of U_M

$$V(\mathbf{w}_j \mid \rho) = \{\mathbf{x} \in U_M \mid \rho(\mathbf{w}_j \mid \mathbf{x}) \geq \rho\} \Leftrightarrow \begin{cases} V(\mathbf{w}_j \mid \rho) = \{\mathbf{x} \in U_M \mid dis(\mathbf{x}, \mathbf{w}_j) \leq d_V(\mathbf{w}_j \mid \rho)\} \\ d_V(\mathbf{w}_j \mid \rho) = M(1-\rho) - s(\mathbf{w}_j) \end{cases}. \tag{16}$$

Note that in Definition 4 we made use of Equations 9 and 11. We call the quantity $d_V(\mathbf{w}_j|\rho)$ the *radius of the match (vigilance) region*. It stands for the maximum L_1 distance a pattern $\mathbf{x}$ can have from the category's representation region, so that the category (with template $\mathbf{w}_j$) still passes the VT for a vigilance parameter value of ρ. Based on Definition 4 we can replace the algebraic definition of the VT, as shown in Equation 11, with a geometric one:

Geometric Definition of the Vigilance Test

A FA category j with template $\mathbf{w}_j$ passes the VT with respect to an input pattern $\mathbf{x} \in U_M$ for a particular value ρ of the vigilance parameter, if and only if $\mathbf{x} \in V(\mathbf{w}_j|\rho)$.

One can observe in Equation 16 that the match region radius decreases with increasing category size. In other words, while a category experiences a representation region expansion, its match region decreases in size. This observation and the fact that $d_V(\mathbf{w}_j|\rho)$ can only be positive hints that the VT enforces a maximum category size, which is controlled by the value of ρ.

Property 1
Due to restrictions solely imposed by the VT, for all $\rho\in[0,1]$ a FA category can reach a maximum size of $M(1-\rho)$ under fast learning. In general, for a FA category j with template $\mathbf{w}_j$ it holds that $R(\mathbf{w}_j)\subseteq V(\mathbf{w}_j|\rho)\ \forall\rho\in[0,1]$. Only if the category's size equals the maximum size $M(1-\rho)$, then $R(\mathbf{w}_j)=V(\mathbf{w}_j|\rho)\ \ \forall\rho\in[0,1]$.

In the case, where $R(\mathbf{w}_j)=V(\mathbf{w}_j|\rho)$, patterns outside category j will never pass the VT for any pattern outside its representation region. Also, upon presentation of patterns inside its representation region, due to the learning law in Equation 4, the category will also not get modified. Therefore, if $R(\mathbf{w}_j)=V(\mathbf{w}_j|\rho)$, category j cannot be updated due to any training pattern and has reached its maximum size. We know at this point that the match region always contains the representation region. Also, if for some pattern $\mathbf{x}$ and category with template $\mathbf{w}$ it holds $\rho(\mathbf{w}_j|\mathbf{x})=\rho$, then $\mathbf{x}$ is located on the boundary of the category's match region. In other words, the match region's boundary represents all points, for which the category will barely pass the VT. Illustrations of a category region for a general case of $\mathbf{w}_j$ when $M=2$ is given in Figure 3. The union of both shaded areas constitutes the match region of the category depicted. We state here without proof that for higher dimensionalities of the input space (higher values of M) the match region's boundary is a convex polytope with their axes of

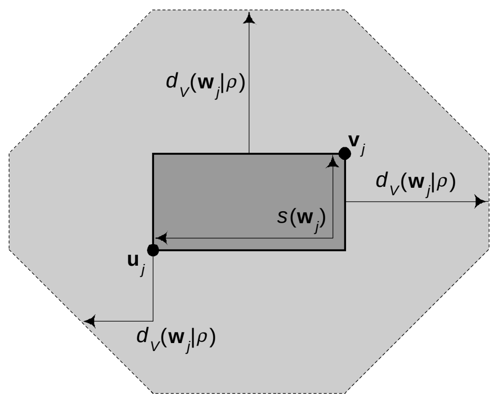

Figure 3: Match region of a template j in 2 dimensions.

symmetry parallel to the ones of the coordinate system. We continue with Property 2, which describes the relationship of a category's match region before and after an update due to a pattern belonging to the category's original match region.

Property 2
During the training phase and using fast learning, the match region of any FA category contracts, whenever the category expands to encode a pattern located inside its match region, but outside the category's representation region. Stated in terms of sets, for any FA category j with template $\mathbf{w}_j$ and any pattern $\mathbf{x}\in V(\mathbf{w}_j|\rho)\text{-}R(\mathbf{w}_j)$ it holds that $V(\mathbf{x}^e\wedge\mathbf{w}_j|\rho)\subset V(\mathbf{w}_j|\rho)\ \forall\rho\in[0,1]$. Also, it holds that $V(\mathbf{x}^e\wedge\mathbf{w}_j|\rho)=V(\mathbf{w}_j|\rho)$, if and only if $\mathbf{x}\in R(\mathbf{w}_j)$. As a general statement, if $\mathbf{x}\in V(\mathbf{w}_j|\rho)$, then $V(\mathbf{x}^e\wedge\mathbf{w}_j|\rho)\subseteq V(\mathbf{w}_j|\rho)$ $\forall\rho\in[0,1]$.

Since match regions are contracting when their related representation regions expands, an immediate result of Property 2 is the following:

Property 3
During the training phase and using fast learning, the match region's hyper-volume of any FA category decreases, whenever the category expands to encode a pattern located inside its match region, but outside the category's representation region, i.e., if $\mathbf{x}\in V(\mathbf{w}_j|\rho)\text{-}R(\mathbf{w}_j)$, then $Vol(V(\mathbf{x}^e\wedge\mathbf{w}_j|\rho))<Vol(V(\mathbf{w}_j|\rho))$.

An example of a 2-dimensional match region contracting is shown in Figure 4 illustrated in the next page, where a representation region expands due to category's j update and its match region decreases in volume (surface, in 2 dimensions), while it remains contained in the original match region. A similar statement to Property 2 and 3 also holds for slow learning and implies that as long as a category is being updated, the match region contracts and decreases in hyper-volume. As an immediate result of Property 2, we conclude the following statement:

Property 4

A category that does not pass the VT for a pattern **x** and a specific value of the vigilance parameter ρ will never pass the VT in future list presentations of off-line FA/FAM training for the same pattern **x** and value of ρ.

So far we have highlighted many aspects of the match region, which relate directly to the notion of the VT. A similar development as in Equation 16 can be performed for the CT.

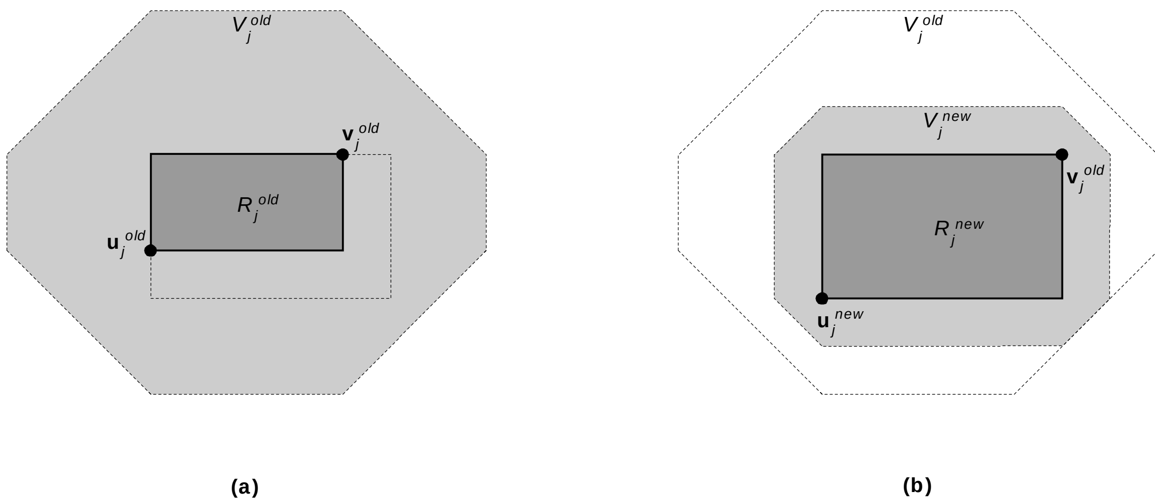

Figure 4: Contraction of match region in 2 dimensions.

Definition 5

We define as *choice (commitment) region* $C(\mathbf{w}_j|a)$ of a category j with template $\mathbf{w}_j$ for a particular value a of the choice parameter the subset of U_M

$$C(\mathbf{w}_j \mid a) = \{\mathbf{x} \in U_M \mid T(\mathbf{w}_j \mid \mathbf{x}) \geq T_u\} \quad \Leftrightarrow \quad \begin{cases} C(\mathbf{w}_j \mid a) = \{\mathbf{x} \in U_M \mid dis(\mathbf{x}, \mathbf{w}_j) \leq d_C(\mathbf{w}_j \mid a)\} \\ d_C(\mathbf{w}_j \mid a) = \dfrac{M+a}{2M+a}\left[\dfrac{M^2}{M+a} - s(\mathbf{w}_j)\right] \end{cases} . \tag{17}$$

In other words, $C(\mathbf{w}_j|a)$ stands for all points of the input space, for which the category j with template $\mathbf{w}_j$ would satisfy the CT, when the choice parameter equals a. Points, for which $T(\mathbf{w}_j|\mathbf{x})=T_u$, lie on the boundary of the choice region. We can also describe the CT using geometrical concepts, as we did with the VT.

Geometric Definition of the Commitment Test

A FA category j with template $\mathbf{w}_j$ passes the CT with respect to a pattern $\mathbf{x} \in U_M$ for a particular value a of the choice parameter if and only if $\mathbf{x} \in C(\mathbf{w}_j|a)$.

In a similar fashion, the quantity $d_C(\mathbf{w}_j|a)$ in Equation 17 is called the *radius of the choice (commitment) region*. Observations similar to the ones that we have stated for the match region radius can be stated for $d_C(\mathbf{w}_j|a)$ as well.

Property 5

Due to restrictions solely imposed by the CT, for all $a \in (0,\infty)$ and a training set of finite cardinality, the least upper bound for any FA category's size equals $M^2/M+a$. Thus, for a FA category j with template $\mathbf{w}_j$ under the same conditions it holds $R(\mathbf{w}_j) \subset C(\mathbf{w}_j|a)$.

For a 2-dimensional category its choice region would resemble in shape to the match region depicted in Figure 3. This is because Equations 16 and 17 are of the same general form. Unfortunately, for the choice region there is no counterpart to Property 2 or 4, but there is a counterpart ot Property 3. Without giving a formal proof, it turns out that after a category has been updated the new choice region does not completely lie within its former choice region. However, a weaker result can be shown as stated below:

Property 6

For all $a \in (0,\infty)$ the choice region of any FA category j decreases in terms of hyper-volume each time the category is updated due to an input pattern inside its choice region, but outside its representation region. In other words, if $\mathbf{x} \in C(\mathbf{w}_j|a)-R(\mathbf{w}_j)$, then $Vol(C(\mathbf{x}^e \wedge \mathbf{w}_j|a)) < Vol(C(\mathbf{w}_j|a))$.

Again, an example of the above statement in 2 dimensions is given in Figure 5, which illustrates the fact that, although the choice region decreases in hyper-volume (surface, in 2 dimensions), it is not completely contained in the category's original choice region. So far we have examined both the VT and the CT separately and the results imply that both perform analogous functionality: they regulate which and how many points are allowed to be selected by a particular category during node competition. To summarize, as the category expands due to new input patterns during the training phase and its size increases, the hyper-volume of both regions decreases.

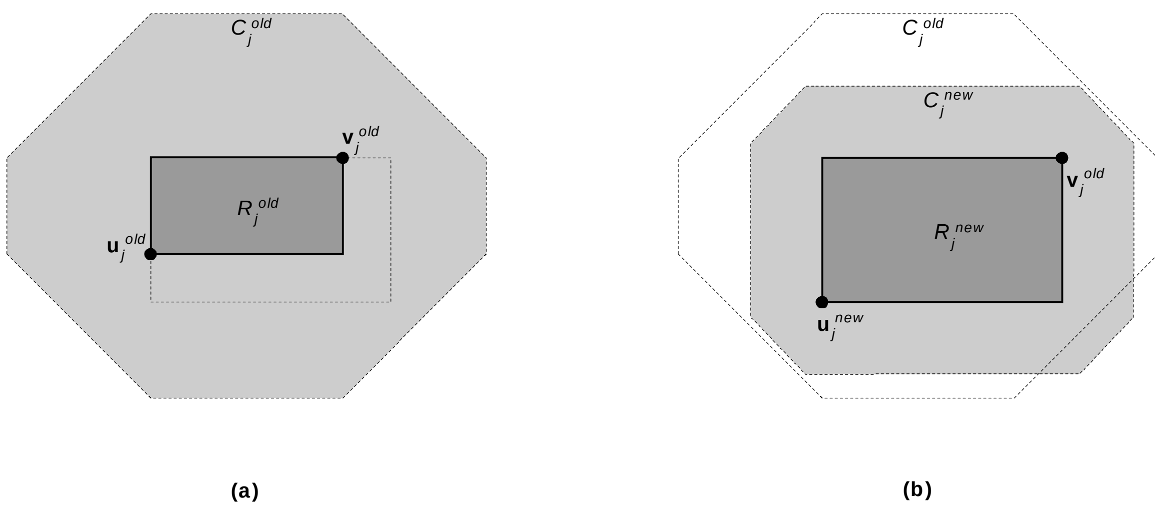

Figure 5: Hyper-volume decrease of choice region in 2 dimensions.

A category enters the candidate set to compete for a particular pattern, if it passes both tests, as stated in Definition 2. Based upon our previous definitions of category regions and our last comment we are led to the following region definition:

Definition 6

We define as *claim region* $L(\mathbf{w}_j|\rho,a)$ of a FA category j with template $\mathbf{w}_j$ for a particular value ρ and a of the vigilance and choice parameter respectively the following subset of U_M

$$L(\mathbf{w}_j \mid \rho, a) = V(\mathbf{w}_j \mid \rho) \cap C(\mathbf{w}_j \mid a) \Leftrightarrow \begin{cases} L(\mathbf{w}_j \mid \rho, a) = \{\mathbf{x} \in U_M \mid dis(\mathbf{x}, \mathbf{w}_j) \le d_L(\mathbf{w}_j \mid \rho, a)\} \\ d_L(\mathbf{w}_j \mid \rho, a) = \min\{d_V(\mathbf{w}_j \mid \rho), d_C(\mathbf{w}_j \mid a)\} \end{cases}. \quad (18)$$

As expected, the quantity $d_L(\mathbf{w}|\rho,a)$ is called the *radius of the claim region*, which also decreases, when a category's size increases. From Figures 4 and 5 we have observed that both the match and the choice region have a similar shape; in general, the regions differ only in their radii. Also, in view of Definition 6 we expect that the claim region will coincide either with the match or the choice region depending on the values of ρ and a.

Property 7

The claim region of a FA category j with template $\mathbf{w}_j$ coincides either with the category's match region or its choice region depending on the value of the vigilance parameter ρ, the value of the choice parameter a and, under certain circumstances, on the category's size $s(\mathbf{w}_j)$. In more detail, for $a \in (0,\infty)$ we discriminate 3 major cases:

i) If $0 \le \rho \le a/(M+a)$, then $L(\mathbf{w}_j|\rho,a)=C(\mathbf{w}_j|a)$.

ii) If $a/(M+a)<\rho<(M+a)/(2M+a)$ and we define $s_{thres}=(2M+a)(1-\rho)-M$, then

- iia) if $s(\mathbf{w}_j)<s_{thres}$, then $L(\mathbf{w}_j|\rho,a)=C(\mathbf{w}_j|a)$.
- iib) if $s_{thres}<s(\mathbf{w}_j)$, then $L(\mathbf{w}_j|\rho,a)=V(\mathbf{w}_j|\rho)$.
- iic) if $s(\mathbf{w}_j)=s_{thres}$, then $L(\mathbf{w}_j|\rho,a)= C(\mathbf{w}_j|a)= V(\mathbf{w}_j|\rho)$.

iii) If $(M+a)/(2M+a) \le \rho \le 1$, then $L(\mathbf{w}_j|\rho,a)=V(\mathbf{w}_j|\rho)$.

An immediate result stemming from Definition 6, Properties 2 and 5 is the following:

Property 8

For all $\rho \in [0,1]$ and $a \in (0,\infty)$ the claim region of any FA category j decreases in terms of hyper-volume each time the category is updated due to an input pattern inside its claim region, but outside its representation region. In other words, if $\mathbf{x} \in L(\mathbf{w}_j|\rho,a)-R(\mathbf{w}_j)$, then $Vol(L(\mathbf{x}^e \wedge \mathbf{w}_j|\rho,a))< Vol(L(\mathbf{w}_j|\rho,a))$.

5. RESULTS

The category regions along with their properties that we've presented so far are sufficient to describe under what conditions a category will be eligible to select a particular pattern during the training and performance phase of FA/FAM. All the results of this section apply for FA modules with parameters $\rho \in [0,1]$ and $a \in (0,\infty)$. The following theorem comes as an immediate result from Property 7:

Theorem 1

Upon presentation of pattern $\mathbf{x}$ during training for any $\gamma \in (0,1]$ or during the performance phase of a FA/FAM network, in order to determine if a particular FA category j of template $\mathbf{w}_j$ enters the candidate set, it suffices to perform only one of the two tests (VT, CT). The test necessary to be performed depends on the values of the network parameters.

i) If $0 \le \rho \le a/(M+a)$, then it suffices to perform only the CT.

ii) If $a/(M+a)<\rho<(M+a)/(2M+a)$ and we define $s_{thres}=(2M+a)(1-\rho)-M$, then

- iia) if $s(\mathbf{w}_j)<s_{thres}$, then it suffices to perform only the CT.
- iib) if $s_{thres}<s(\mathbf{w}_j)$, then it suffices to perform only the VT.
- iic) if $s(\mathbf{w}_j)=s_{thres}$, then perform either the CT or the VT.

iii) If $(M+a)/(2M+a) \le \rho \le 1$, then it suffices to perform only the VT.

Definition 7

We define as *a-dominant* region the subset $0 \le \rho \le a/(M+a)$ of the (ρ, a) parameter space $[0,1] \times (0,\infty)$.

The vigilance ρ is used in FA/FAM only in the VT. In case (i) of Theorem 1 we notice that, if for a category the CT is satisfied with respect to a certain pattern, then the corresponding VT will automatically be satisfied as well. This fact leads us to the following result:

Corollary 1.1

If some FA module, which is part of a FA/FAM network, operates in the a-dominant region of the (ρ, a) parameter space, then the training and performance phase does not depend on the value of ρ.

Corollary 1.1 tells us, for example, that, if a FA network operates in the a-dominant region, the number of categories it is going to create during training does not depend on ρ. An immediate result derived from Property 7 is the following theorem pertaining to the maximum size of categories.

Theorem 2

For a FA/FAM network that has been trained with a finite cardinality training set the size of FA categories is limited by the following rules

i) If $0\leq\rho<(M+a)/(2M+a)$, then the category size has a least upper bound of $s(\mathbf{w})<M^2/(M+a)$.

ii) If $(M+a)/(2M+a)\leq\rho\leq 1$, then $s(\mathbf{w})\leq M(1-\rho)$.

Both statements can be combined in a single inequality

$$s(\mathbf{w}) \leq M\left[1-\max\left\{\rho, \frac{a}{M+a})\right\}\right]. \tag{19}$$

The previous theorem refines somehow an older result[1] that stated $s(\mathbf{w})\leq M(1-\rho)$ for all values of $\rho\in[0,1]$ and $a\in[0,\infty)$. In our last figure featured in the next page, Figure 6, different regions of the (ρ, a) plane are pointed out and related to the results of this section. The curves graphed are $\rho=\phi_1(a)=a/(M+a)$ and $\rho=\phi_2(a)=(M+a)/(2M+a)$. In Figure 6a the area above $\rho=\phi_2(a)$ represents choices of the parameters, for which the satisfaction of the VT implies the satisfaction of the CT for all categories with respect to any input pattern. The contrary holds for choices of (ρ ,a) below the curve $\rho=\phi_1(a)$. The shaded area signifies the parameter choices, where the necessity of the tests depends on the size of each category. In Figure 6b the curve $\rho=\phi_2(a)$ divides the plane into two different areas: for a choice of (ρ, a) pairs below $\rho=\phi_2(a)$ only a least upper bound exists for the size of categories and for pairs above $\rho=\phi_2(a)$ the categories can reach a maximum size of $M(1-\rho)$.

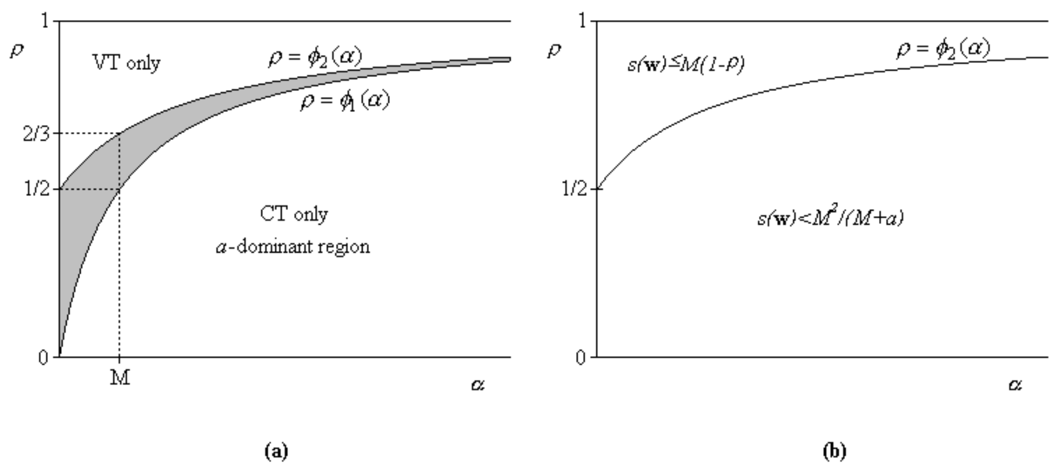

Figure 6: Regions of interest in the (ρ,a) parameter plane.

6. CONCLUSIONS

In this paper we have highlighted the commitment test as a category-filtering device similar to the vigilance test. The two tests conjointly determine the eligibility of nodes to compete for presented patterns in FA/FAM training or performance phase. Additionally, their geometric aspect in the form of category regions has helped us in understanding their functionality and behavior especially during training. Based on the existence of these regions we were led to a few results concerning FA/FAM, which are primarily of theoretical interest. From a practical perspective, the most interesting result pertains to the existence of a-dominant region of the (ρ, a) parameter space. FA/FAM behavior does not depend on ρ, when the network parameter selection is a-dominant. Thus, a researcher experimenting with different parameter values on a FA/FAM network needs only to consider distinct values of a, when (ρ, a) belongs to the a-dominant region and distinct values of pairs (ρ, a) only if otherwise. Finally, the above category regions and produced results can also be adapted with only a few modifications to the case of Hypersphere-ART[6] and Ellipsoid-ART[7] categories, which use hyper-spheres and hyper-ellipsoids instead of hyper-rectangles for category representation.

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