Ellipsoid ART/ARTMAP Category Regions for the Choice-by-Difference Category Choice Function

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ABSTRACT

In the recent past category regions have been introduced as new geometrical concepts and provide a visualization tool that facilitates significant insight into the nature of the competition among categories during both the training and performance phase of Fuzzy ART (FA) and Fuzzy ARTMAP (FAM). These regions are defined as the geometric interpretation of the Vigilance Test and the competition of each category with an uncommitted F_2-layer node for a specific input pattern (Commitment Test). In this paper we show how the notion of category regions can be naturally extended to Ellipsoid ART (EA) and Ellipsoid ARTMAP (EAM) and focus on the regions' theoretical properties, when considering the Choice-by-Difference category choice function. Based on these properties we state three theoretical results applicable to both EA and EAM. Specifically, if ρ and a denote the vigilance and the choice parameter respectively, we show that in certain areas of the (a,ρ) plane the result of EA/EAM training is independent of the specific value of either ρ or ω (parameter of the activation function value for an uncommitted F_2-layer node). Finally, we provide a refined upper bound on the size of categories created in EA/EAM during training. All the results are immediately applicable to FA/FAM as well.

Keywords: category regions, Ellipsoid ART, Ellipsoid ARTMAP, adaptive resonance theory, Choice-by-Difference

1. INTRODUCTION

Ellipsoid ART[1,2] (EA) and *Ellipsoid ARTMAP*[1,2] (EAM) are two neural network architectures that follow the *adaptive resonance theory* (ART) paradigm[3] introduced by Grossberg and have been recently introduced as alternatives to Fuzzy-ART[4] (FA) and Fuzzy-ARTMAP[5] (FAM). A major function of ART architectures is to group similar input patterns into *categories*. These categories are constructed in a self-organizing fashion and constitute the building block of knowledge/memory representation for all ART architectures. While FA and FAM utilize categories whose geometric representations are axis-parallel hyper-rectangles, EA and EAM employ arbitrarily oriented hyper-ellipsoids for the same purpose. EA is used for unsupervised clustering tasks and EAM, which consists of two EA modules interconnected via an inter-ART module, is capable of learning associative maps between an input and an output domain in a supervised manner. When its output domain coincides with a set of class labels, EAM can perform classification tasks (EAM classifier). Despite the different category representation shape, due to their design, EA and EAM share almost all properties of learning with their fuzzy counterparts. First, they are both capable of off-line (batch) and on-line (incremental) learning. Under fast learning rule[1,2,4] assumptions, both exhibit fast, stable and finite learning: the networks' knowledge stabilizes relatively fast after a finite number of *list presentations* (epochs). Furthermore, they both feature novelty detection mechanisms that identify input patterns not typical in relation to previously experienced inputs. Also, due to the specifics of their neural architecture, responses of EA and EAM to specified inputs are easily explained, in contrast to other neural network models, where in general it is difficult to explain why an input pattern **x** produced an output **y**.

We must note that significant insight into the functionality of FA/FAM has been gained by studying certain geometrical concepts related to FA categories. For example, the fact that FA categories can be represented as hyper-rectangles in the input domain has aided in the development of several properties of learning and has increased our level of understanding about the inner workings of these architectures. In this paper we define EA category regions based on the use of Choice-by-Difference category choice function and examine their properties. In the sequel, these properties

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will lead us to the discovery that, depending on the network parameter values of EA/EAM, for all already created EA categories and for all input patterns presented, satisfaction of the VT simultaneously implies satisfaction of the CT and vice versa. Also, pertaining to EA/EAM's training phase, we will derive upper bounds for EA category sizes.

The rest of the paper is organized as follows. In Section 2 we provide some background on EA and EAM, such as basic elements and functionality of these two architectures. Next, in Section 3 we continue with the definitions of category regions and present some of their important properties. Based on these properties in Section 4 we derive 3 theoretical results that are applicable to both EA and EAM. Finally, in Section 5 we summarize all our findings and underline the importance of the derived results.

2. ELEMENTS OF ELLIPSOID ART/ARTMAP

A block diagram of an EA module is presented in Figure 1. The module consists of two subsystems: *attentional* and *orienting*. Input patterns are presented to the F_1 layer of the attentional subsystem. Layers F_1 and F_2 are interconnected via top-down and bottom-up connections and each one of these connections feature a weight. In specific, the vector $\mathbf{w}_j$ consisting of all top-down weights emanating from F_2-layer node j is called *template* of j. The F_2-layer itself is a MAXNET featuring inhibitory lateral connections and consists of two types of nodes: *committed* and *uncommitted.* Committed nodes feature a template that contains the description of a single *EA category* (cluster) that has been learned via training and which summarizes the patterns that have been encoded by the corresponding F_2-layer node. In contrast, uncommitted nodes do not correspond to real categories and constitute the blank memory of the system.

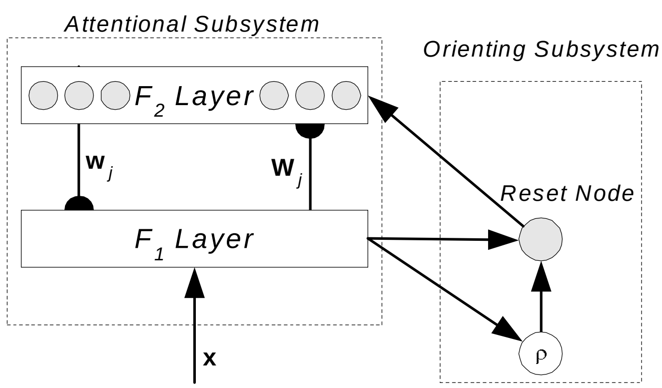

Figure 1: Block diagram of an Ellipsoid ART module.

On the other hand, EAM is comprised of two independent EA modules, ART_a and ART_b, bridged via an inter-ART module. This last module encodes the associations between input domain and output domain EA categories learned by ART_a and ART_b respectively. Erroneous category associations are corrected via the *match tarcking*[1,2] mechanism. For classification tasks the role of the ART_b module in EAM becomes trivial. The output domain is the set of class labels pertinent to the classification problem at hand and ART_b will create a distinct EA category for each class label. Since EAM finds more applications as a classifier (as FAM does), in the future, when we will refer to EAM and its network parameters, we will mean the EAM classifier and its ART_a module's network parameters.

The geometric representation of an EA category (or simply, category) is a hyper-ellipsoid embedded in the input domain of the EA module. It is the description of this hyper-ellipsoid that is stored in the category's template. If M is the input domain dimensionality, for a committed node j its template $\mathbf{w}_j$ is expressed as $\mathbf{w}_j=[\mathbf{m}_j\ \mathbf{d}_j\ Rj]$, where $\mathbf{m}_j \in \mathrm{R}^M$ is the hyper-ellipsoid's *center location vector,* $\mathbf{d}_j \in \mathrm{R}^M$ is its *orientation vector* and R_j is its (Mahalanobis) *radius*. The last quantity also represents the length of the hyper-ellipsoid's major semi-axis. A characteristic of EA categories is that the length of the rest of the semi-axes always equals μR_j, where $\mu \in (0,1]$ is the EA module's *ratio parameter* and which is common to all categories. The aforementioned constraint arises from some design requirements of EA/EAM[1,2]. Also, if $\mu=1$, EA/EAM simplify to Hyper-sphere ART[6] and Hyper-sphere ARTMAP[6] respectively. A depiction of a 2-

dimensional EA category is provided in Figure 2a. We define as the size $s(\mathbf{w}_j)$ of category j and the distance $dis(\mathbf{x},\mathbf{w}_j)$ of a real-valued input pattern **x** from category j the quantities

$$s(\mathbf{w}_j) = 2R_j \,. \tag{1}$$

$$dis(\mathbf{x}, \mathbf{w}_j) = \max\left\{ \left\| \mathbf{x} - \mathbf{m}_j \right\|_{\mathbf{C}_j}, R_j \right\} - R_j \,. \tag{2}$$

where

$$\left\| \mathbf{x} - \mathbf{m}_j \right\|_{\mathbf{C}_j} = \frac{1}{\mu} \sqrt{ \left\| \mathbf{x} - \mathbf{m}_j \right\|_2^2 - (1-\mu^2) \left[\mathbf{d}_j^{\,T} (\mathbf{x} - \mathbf{m}_j) \right]^2 } \,. \tag{3}$$

The quantity in Equation 3 is a weighted Euclidian distance measure; more precisely, it is the Mahalanobis distance of **x** from the center $\mathbf{m}_j$ with weight matrix $\mathbf{C}_j = \mathbf{C}_j^T = \frac{1}{\mu^2}\left[\mathbf{I} - (1-\mu^2)\mathbf{d}_j \mathbf{d}_j^{\,T}\right]$, where **I** is the identity matrix. Also, in Equation 3 $\left\| \mathbf{x} - \mathbf{m}_j \right\|_2$ stands for the standard L_2-norm (Euclidian) distance of **x** from $\mathbf{m}_j$. A pattern **x**, for which $dis(\mathbf{x},\mathbf{w}_j)=0$, is considered to be already encoded (learned) by category j. In Figure 2a the shaded area represents the set of all input domain patterns that j already encodes and are located inside the category's ellipsoid. At this point we note that templates of F_2-layer uncommitted nodes are not defined and that there is no geometric representation for their corresponding "virtual" categories.

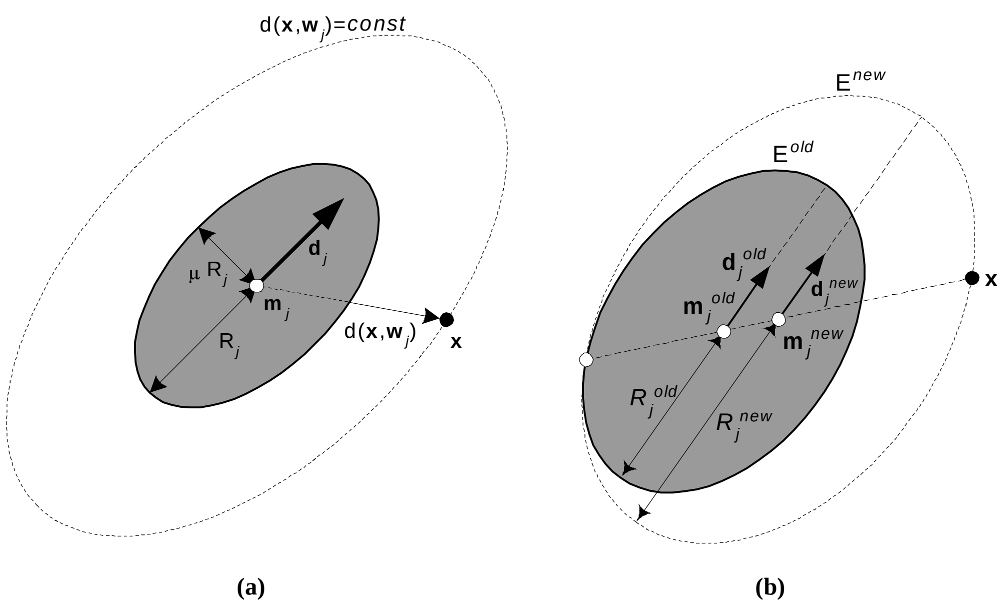

Figure 2: Geometric representation (a) and update (b) of an EA category in 2 dimensions.

EA/EAM have two modes of operation: *training phase* and *performance phase*. Both phases are of similar nature except that during performance phase no category updates or creations take place. During training phase EA/EAM incrementally clusters input domain data into categories by committing F_2-layer nodes and by updating appropriately their templates. Before any learning takes place, all F_2-layer nodes are uncommitted. As more knowledge is accumulated about the input domain during training, F_2-layer nodes become gradually committed. An integral part of both the training and performance phases of EA/EAM is the competition among F_2-layer nodes upon the presentation of an input pattern **x**. Nodes compete in terms of *category choice function* (CCF – or simply, *activation function*) value $T(\mathbf{w}_j|\mathbf{x})$. The node of the highest CCF value and smallest index is declared as being the winner of the competition. Uncommitted nodes also participate in the competition with a parameterized, but constant, CCF value of T_u for all patterns **x**. Upon completion of identifying the winning node J, its *category match function* (CMF) value $\rho(\mathbf{w}_J|\mathbf{x})$ is being calculated. The CMF for a committed node j is given in Equation 4, while for an uncommitted node j it is defined to be constant $\rho(\mathbf{w}_j|\mathbf{x})=1$ for all patterns $\mathbf{x}$.

$$\rho(\mathbf{w}_j \mid \mathbf{x}) = \frac{D - s(\mathbf{w}_j) - dis(\mathbf{x}, \mathbf{w}_j)}{D} \quad \overset{Eq.1,2}{\Rightarrow} \quad \rho(\mathbf{w}_j \mid \mathbf{x}) = 1 - \frac{R_j + \max\left\{ \left\| \mathbf{x} - \mathbf{m}_j \right\|_{\mathbf{C}_j}, R_j \right\}}{D}. \tag{4}$$

where $D>0$ is another network parameter and whose value is large enough so that $\rho(\mathbf{w}_j|\mathbf{x})\geq 0$ for all categories and all input patterns. Next, the CMF value $\rho(\mathbf{w}_J|\mathbf{x})$ of J is compared to the vigilance network parameter value $\rho\in[0,1]$. This comparison, which is called *vigilance test* (VT), checks if the following inequality holds:

$$\rho(\mathbf{w}_J \mid \mathbf{x}) \geq \rho. \tag{5}$$

If Equation 5 is not satisfied, then J is temporarily reset (*mismatch reset* – its CCF value is set to zero) until the presentation of the next input pattern. With J effectively excluded from the competition, the search continues for the category featuring the highest CCF value among the remaining ones. If all categories become gradually reset, an uncommitted node will eventually emerge as the winning node; then we say that $\mathbf{x}$ choses an uncommitted node. On the other hand, if Equation 5 is true, we say that pattern $\mathbf{x}$ chooses node (category) J. EA/EAM's performance phase stops right at this point, where $\mathbf{x}$ chooses a node. If EA is operating in training phase, the template of the winning node is going to be updated. In EAM's training phase, however, an extra step is involved. Only if J predicts the correct class label of $\mathbf{x}$, J will be updated. Otherwise, match tracking goes into effect, J becomes temporarily reset until th enext pattern presentation and the competition process repeats itself until a suitable winning category is found. More details about match tracking can be found in the references[1,2,5]. At this point let us assume that J is about to be updated. If J is an uncommitted node, $\mathbf{x}$ will initiate the creation of a new category with the following template elements:

$$\begin{aligned} \mathbf{m}_J &= \mathbf{x} \\ \mathbf{d}_J &= \mathbf{0} \cdot \\ R_J &= 0 \end{aligned} \tag{6}$$

A category of this kind and any other category j with $s(\mathbf{w}_j)=0$ is called *point category*. If J is a committed node with template $\mathbf{w}_J^{old}=[\mathbf{m}_J^{old}\ \mathbf{d}_J^{old}\ R_J^{old}]$, its updated template elements are given by Equation 7 and 8.

$$\mathbf{m}_J^{new} = \mathbf{m}_J^{old} + \frac{\gamma}{2} dis(\mathbf{x}, \mathbf{w}_J) \frac{\left(\mathbf{x} - \mathbf{m}_J^{old}\right)}{\left\| \mathbf{x} - \mathbf{m}_J^{old} \right\|_{C_J^{old}}} \quad \overset{Eq.1,2}{\Rightarrow} \quad \mathbf{m}_J^{new} = \mathbf{m}_J^{old} + \frac{\gamma}{2}\left(1 - \frac{\min\left\{ R_J^{old}, \left\| \mathbf{x} - \mathbf{m}_J^{old} \right\|_{C_J^{old}} \right\}}{\left\| \mathbf{x} - \mathbf{m}_J^{old} \right\|_{C_J^{old}}} \right)\left(\mathbf{x} - \mathbf{m}_J^{old}\right). \tag{7}$$

$$R_J^{new} = R_J^{old} + \frac{\gamma}{2} dis(\mathbf{x}, \mathbf{w}_J) \quad \overset{Eq.1,2}{\Rightarrow} \quad R_J^{new} = R_J^{old} + \frac{\gamma}{2}\left(\max\left\{ R_J^{old}, \left\| \mathbf{x} - \mathbf{m}_J^{old} \right\|_{C_J^{old}} \right\} - R_J^{old} \right). \tag{8}$$

The network parameter $\gamma\in(0,1]$ is called *learning rate*. In the special case where $\gamma=1$, we say that *fast learning* is being performed; for any other value $\gamma<1$, *slow learning*. The orientation vector is updated only if J is a point category and $\mathbf{x}$ does not coincide with its center location, as shown in Equation 9.

$$\mathbf{d}_J = \frac{\mathbf{x} - \mathbf{m}_J^{old}}{\left\| \mathbf{x} - \mathbf{m}_J^{old} \right\|_2} \quad \mathbf{x} \neq \mathbf{m}_J^{old}. \tag{9}$$

When J is no longer a point category, its orientation vector $\mathbf{d}_J$ will remain unchanged. Figure 2b shows the update of a category's template elements for fast learning. Due to pattern $\mathbf{x}$ the category's initial representation region (ellipsoid) will expand from E^{old} to E^{new} so that it will barely include $\mathbf{x}$ into E^{new}. After a category has been updated or created, training proceeds with the presentation of the next input pattern and so forth. A complete presentation of the entire training set is called a *list presentation* (*epoch*). When using fast learning we say that EA has completed its learning task (converged), when after a complete list presentation no categories were updated or created.

In the original presentation of EA/EAM[1,2] the CCF, which is was used is of *Weber Law*[1,2] type:

$$T(\mathbf{w}_j \mid \mathbf{x}) = \frac{D - s(\mathbf{w}_j) - dis(\mathbf{x}, \mathbf{w}_j)}{D - s(\mathbf{w}_j) + a} \quad \overset{Eq.1,2}{\Rightarrow} \quad T(\mathbf{w}_j \mid \mathbf{x}) = \frac{D - R_j - \max\left\{ \left\| \mathbf{x} - \mathbf{m}_j \right\|_{\mathbf{C}_j}, R_j \right\}}{D - 2R_j + a}. \tag{10}$$

where a>0 is the network's *choice parameter*. However, in this paper we are going to equip EA (and EAM) with the C*hoice-by-Difference*[7] (CBD) choice function, which for a committed node j is defined as

$$T(\mathbf{w}_j \mid \mathbf{x}) = D(2-a) - as(\mathbf{w}_j) - dis(\mathbf{x}, \mathbf{w}_j) \quad \overset{Eq.1,2}{\Rightarrow} \quad T(\mathbf{w}_j \mid \mathbf{x}) = D(2-a) - (2a-1)R_j - \max\left\{ \left\| \mathbf{x} - \mathbf{m}_j \right\|_{\mathbf{C}_j}, R_j \right\}. \tag{11}$$

and for an uncommitted node j as

$$T_u = D\left[1 - 2(1-a)(\omega - 1)\right]. \tag{12}$$

where $\omega \geq 1/2$ is a network parameter that controls the competitiveness of uncommitted nodes. The higher the value of ω, the less competitive the uncommitted nodes become. Equation 12 shows that the CCF value of uncommitted nodes features a constant value for all input patterns. Note also, that for the CBD choice function it is assumed that $a \in (0,1)$. From Equation 11 we can show that for all categories j and input patterns $\mathbf{x}$ it holds

$$T_{\min} = D(1-a) \leq T(\mathbf{w}_j \mid \mathbf{x}) \leq D(2-a) = T_{\max}. \tag{13}$$

From Equation 13 we can conclude that, if ω <1/2, EA/EAM becomes unstable, because $T_u > T_{max}$ and, thus, input patterns always select uncommitted nodes. In general, the comparison of CCF values to T_u is termed as *commitment test*[8] (CT). The CT is satisfied if

$$T(\mathbf{w}_j \mid \mathbf{x}) \geq T_u. \tag{14}$$

Non-satisfaction of Equation 14 means that $\mathbf{x}$ will choose an uncommitted node over node j. The CT is implicitly performed during the node competition for pattern selection that we have described earlier. In the past it has been shown[8] that the CT, i.e., the competition against uncommitted nodes, is a category-filtering mechanism similar (but not identical) to the VT. The fact that a category j does not satisfy the VT and/or the CT for a specific input pattern $\mathbf{x}$ can be interpreted as "$\mathbf{x}$ does not match the characteristics of j and therefore should not select j". In essence, both the CT and the VT act as novelty detection mechanisms that implement match-based learning and that are able to detect patterns that are atypical with respect to whatever input has been experienced in the past by an EA/EAM network.

3. CATEGORY REGIONS INDUCED BY CHOICE-BY-DIFFERENCE

The notion of category regions surfaces upon investigation of the VT's and CT's geometric interpretation. Given a particular category j of known template $\mathbf{w}_j$ we want to identify the regions of the EA module's input domain (the set of input patterns) for which j satisfies the VT and/or the CT. The introduction of category regions for FA categories[8], when Weber Law CCF is used, resulted to a better geometrical understanding of under which conditions a specific category has the potential of being chosen by an input pattern. Furthermore, via the study of the region's properties it was possible to show certain theoretical results[8] that pertain to both the training and performance phase of FA/FAM. By virtue of EA/EAM's design, it can be shown that all of these results also hold for EA/EAM with slight modifications, when Weber Law CCF is used. In this paper we are going to study the category regions defined by the CBD choice function for EA categories. All the definitions and properties that we will state can be extended to FA/FAM in a straightforward manner. In this particular section we present the category regions' definitions and a collection of pertinent properties. Due to lack of space we omit the proofs of these properties, but hopefully the reader will be able to visually verify them in the 2-dimensional case with the aid of the figures and comments provided in this section. Also, note that all the definitions and properties presented in this section are valid for any value $\mu \in (0,1]$.

Definition 1

We define as *representation region* $R(\mathbf{w}_j)$ of a category (committed node) j with template $\mathbf{w}_j$ the following subset of R^M

$$R(\mathbf{w}_j) = \left\{ \mathbf{x} \in R^M \mid dis(\mathbf{x}, \mathbf{w}_j) = 0 \right\} \quad \overset{Eq.1,2}{\Rightarrow} \quad R(\mathbf{w}_j) = \left\{ \mathbf{x} \in R^M \mid \left\| \mathbf{x} - \mathbf{m}_j \right\|_{\mathbf{C}_j} \leq R_j \right\}. \tag{15}$$

In the example of Figure 2b the category's representation region would be the shaded area. From Equation 7 and 8 it can be shown that if $\mathbf{x}$ chooses j and $\mathbf{x} \in R(\mathbf{w}_j)$, then no category update will occur. Also, from Figure 2b it becomes evident that, if j has initially a template of $\mathbf{w}_j^{old}$ and $\mathbf{x} \notin R(\mathbf{w}_j^{old})$, then after j's template update to $\mathbf{w}_j^{new}$ via Equation 7, 8 (and 9, if applicable) it will hold that $R(\mathbf{w}_j^{old}) \subset R(\mathbf{w}_j^{new})$, otherwise $R(\mathbf{w}_j^{old}) \subset R(\mathbf{w}_j^{new})$. This implies that $R(\mathbf{w}_j^{old}) \subseteq R(\mathbf{w}_j^{new})$ for any $\mathbf{x} \in R^M$, $\gamma \in (0,1]$ and, when a category is being modified, its representation region expands and includes more points of the input domain. Next, we proceed with a definition that adorns the VT with a geometrical interpretation.

Definition 2

We define as *match (vigilance) region* $V(\mathbf{w}_j|\rho)$ of a category (committed node) j with template $\mathbf{w}_j$ for a particular value ρ of the vigilance parameter the following subset of R^M

$$V(\mathbf{w}_j \mid \rho) = \left\{\mathbf{x} \in R^M \mid \rho(\mathbf{w}_j \mid \mathbf{x}) \geq \rho\right\} \overset{Eq.4}{\Rightarrow} \begin{cases} V(\mathbf{w}_j \mid \rho) = \left\{\mathbf{x} \in R^M \mid dis(\mathbf{x}, \mathbf{w}_j) \leq d_V(\mathbf{w}_j \mid \rho)\right\} \\ d_V(\mathbf{w}_j \mid \rho) = M(1-\rho) - s(\mathbf{w}_j) \end{cases}. \tag{16}$$

We call the quantity $d_V(\mathbf{w}_j|\rho)$ the *radius of the match (vigilance) region*. It stands for the maximum weighted L_2 distance a pattern $\mathbf{x}$ can have from the category's representation region, so that the category (with template $\mathbf{w}_j$) still passes the VT for a vigilance parameter value of ρ. Due to Definition 2, $\mathbf{x}\in V(\mathbf{w}_j|\rho)$ if and only if j passes the VT with respect to $\mathbf{x}$. From Equation 16 we observe that the match region radius decreases with increasing category size. When this radius becomes 0, it can be shown that $V(\mathbf{w}_j|\rho)=R(\mathbf{w}_j)$ and category j cannot expand any more. This fact implies that depending on the value of ρ the match region imposes a maximum size for the size of categories.

Property 1

Due to restrictions solely imposed by the VT, during training, for all $\rho\in[0,1]$ and $\gamma\in(0,1]$ an EA category can reach a maximum size of $D(1-\rho)$. Also for an EA category j with template $\mathbf{w}_j$ it holds that $R(\mathbf{w}_j)\subseteq V(\mathbf{w}_j|\rho)\ \forall\rho\in[0,1]$. Only if the category's size equals the maximum size $D(1-\rho)$, then $R(\mathbf{w}_j)=V(\mathbf{w}_j|\rho)\ \ \forall\rho\in[0,1]$.

We know at this point that the match region always contains the representation region. Also, if for some pattern $\mathbf{x}$ and category with template $\mathbf{w}$ it holds $\rho(\mathbf{w}_j|\mathbf{x})=\rho$, then $\mathbf{x}$ is located on the boundary of the category's match region. In other words, the match region's boundary represents all points, for which the category will barely pass the VT. A typical illustration of an EA category's match region in 2 dimensions is given in Figure 3. The union of both ellipsoidal, shaded areas constitutes j's match region. According to what we have presented so far, j will pass the VT with respect to $\mathbf{x}_1$ and $\mathbf{x}_4$ (since $\mathbf{x}_4\in R(\mathbf{w}_j)\subset V(\mathbf{w}_j|\rho)$), it will barely pass the VT for $\mathbf{x}_2$ and it will fail it for $\mathbf{x}_3$ (since $\mathbf{x}_3\notin V(\mathbf{w}_j|\rho)$). It becomes obvious that for higher dimensionalities of the input space the match region's boundary generalizes to a hyper-ellipsoid.

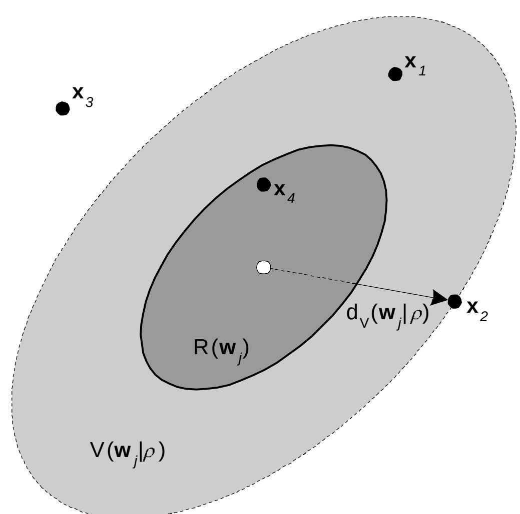

Figure 3: Match region of a template j in 2 dimensions.

Property 2
During the training phase, for any $\rho\in[0,1]$ and $\gamma\in(0,1]$ the match region of any EA category contracts, whenever the category expands due to an update. Stated in terms of sets, for any EA category j with template $\mathbf{w}_j^{old}$ and any pattern $\mathbf{x}\in V(\mathbf{w}_j^{old}|\rho)$-$R(\mathbf{w}_j^{old})$ it holds that $V(\mathbf{w}_j^{new}|\rho)\subset V(\mathbf{w}_j^{old}|\rho)$ $\forall\rho\in[0,1]$ and $\gamma\in(0,1]$. Also, it holds that $V(\mathbf{w}_j^{new}|\rho)= V(\mathbf{w}_j^{old}|\rho)$, if and only if $\mathbf{x}\in R(\mathbf{w}_j^{old})$. As a general statement, if $\mathbf{x}\in$ $\mathbf{x}\in V(\mathbf{w}_j^{old}|\rho)$, then $\mathbf{x}\in V(\mathbf{w}_j^{new}|\rho)\subseteq V(\mathbf{w}_j^{old}|\rho)$ $\forall\rho\in[0,1]$ and $\gamma\in(0,1]$.

Since match regions contract whenever their related representation regions expand, an immediate result of Property 2 is the following:

Property 3
During the training phase, for any $\rho\in[0,1]$ and $\gamma\in(0,1]$ the match region's hyper-volume of any EA category decreases, whenever the category expands due to an update, i.e., if $\mathbf{x}\in V(\mathbf{w}_j^{old}|\rho)$-$R(\mathbf{w}_j^{old})$, then $Vol(V(\mathbf{w}_j^{new}|\rho))<Vol(V(\mathbf{w}_j^{old}|\rho))$.

Figure 4 provides an example that demonstrates the last two properties. In the figure the category's representation region R_j^{old} expands to R_j^{new} due to category's j update (Figure 4a) and its match region decreases in volume (surface, in 2 dimensions), while it remains contained in the original match region V_j^{old} (Figure 4b). Notice that for convenience we have dropped the dependence of the regions from templates or parameters.

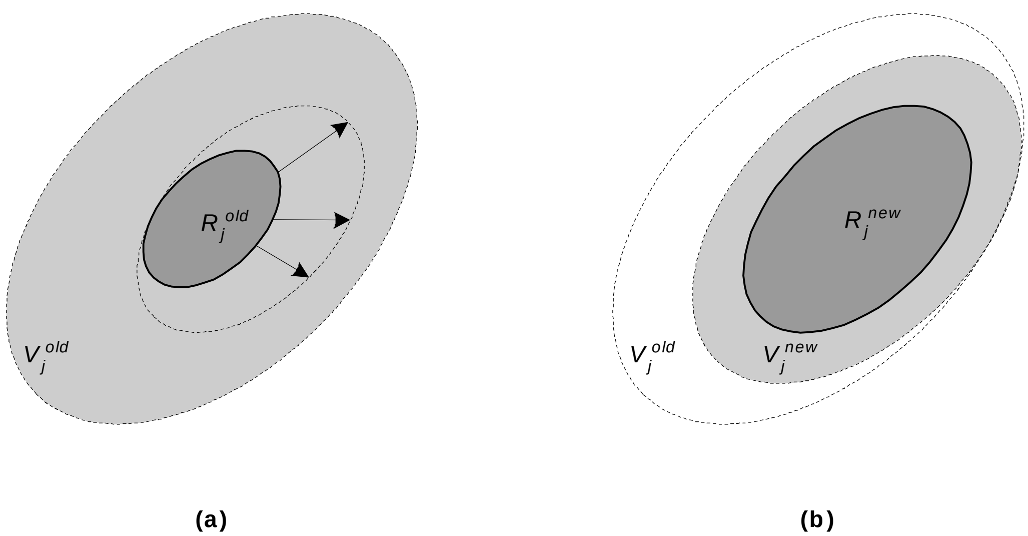

Figure 4: Contraction of match region in 2 dimensions.

A similar study can be performed on the geometric interpretation of the CT by appropriately defining an associated category region and then studying its properties.

Definition 3
For the Choice-by-Difference CCF we define as *choice (commitment) region* $C(\mathbf{w}_j|a,\omega)$ of an EA category j with template $\mathbf{w}_j$ for particular values of the parameters a and ω the subset of R^M

$$C(\mathbf{w}_j \mid a,\omega) = \left\{\mathbf{x}\in R^M \mid T(\mathbf{w}_j \mid \mathbf{x}) \geq T_u\right\} \overset{Eq.11,12}{\Rightarrow} \begin{cases} C(\mathbf{w}_j \mid a) = \left\{\mathbf{x}\in R^M \mid dis(\mathbf{x},\mathbf{w}_j) \leq d_C(\mathbf{w}_j \mid a,\omega)\right\} \\ d_C(\mathbf{w}_j \mid a,\omega) = D(1-a)(2\omega-1) - as(\mathbf{w}_j) \end{cases}. \quad (17)$$

In other words, $C(\mathbf{w}_j|a,\omega)$ stands for all points of the input space, for which the category j with template $\mathbf{w}_j$ would satisfy the CT for the specific values of a and ω. Category j passes the CT for $\mathbf{x}$ if and only of $\mathbf{x}\in C(\mathbf{w}_j|a,\omega)$. Points, for which $T(\mathbf{w}_j|\mathbf{x})=T_u$, lie on the boundary of j's choice region. The quantity $d_C(\mathbf{w}_j|a,\omega)$ in Equation 16 is called the *radius of the choice (commitment) region*. Observations similar to the ones that we have stated for the match region radius can be stated for $d_C(\mathbf{w}_j|a,\omega)$ as well. Before we continue further with properties, let us define the quantities

$$\rho^- = 1-(2\omega-1)(a-1)\,. \tag{18}$$

$$\rho^+ = \frac{a}{(2\omega-1)D+a}\,. \tag{19}$$

It can be easily shown, that $\rho^+ < \rho^- < 1$ for all $a\in(0,1)$ and $\omega\in[1/2,\infty)$. These two quantities will prove helpful in the sequel.

Property 4

Due to restrictions solely imposed by the CT, during training, for all $a\in(0,1)$, $\omega\in[1/2,\infty)$ and $\gamma\in(0,1]$ the least upper bound for any EA category's size equals $D(1-\rho^+) = \frac{D(1-a)(2\omega-1)}{a}$. Moreover, for any EA category j with template $\mathbf{w}_j$ under the same conditions it holds $R(\mathbf{w}_j)\subset C(\mathbf{w}_j|a,\omega)$.

EA choice regions have the same shape as match regions, but differ in radii, as can be observed from Definitions 2 and 3. Let us note that for choice regions there is no counterpart to Property 2 or 4. As it turns out, the choice region after an update does not completely lie within the original choice region (see Figure 5). However, there is a counterpart to Property 3.

Property 5

During the training phase, for all $a\in(0,1)$, $\omega\in[1/2,\infty)$ and $\gamma\in(0,1]$ the choice region of any EA category j decreases in terms of hyper-volume each time the category is being updated due to an input pattern. In other words, if $\mathbf{x}\in C(\mathbf{w}_j^{old}|a,\omega)\text{-}R(\mathbf{w}_j^{old})$, then $Vol(C(\mathbf{w}_j^{new}|a,\omega)) < Vol(C(\mathbf{w}_j^{old}|a,\omega))$.

Figure 5 presents a 2-dimensional demonstration of Property 5. Although the choice region decreases in hyper-volume (surface, in 2 dimensions) from C_j^{old} to C_j^{new}, C_j^{new} is not completely contained within C_j^{old}.

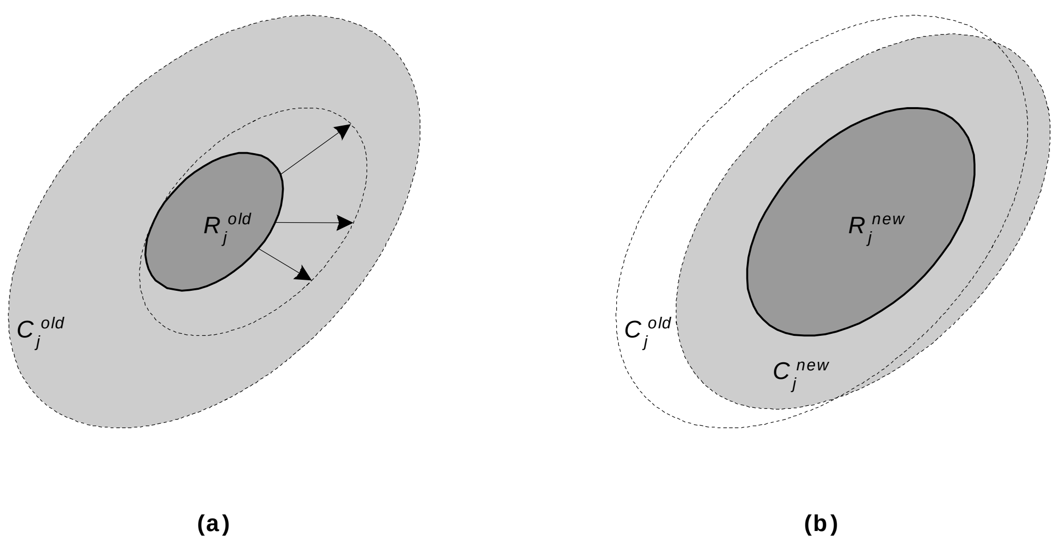

Figure 5: Hyper-volume decrease of choice region in 2 dimensions.

Up to this point we have examined both the VT and the CT separately but we saw that their geometric interpretations are similar, which justifies our statement that both of these tests share a common functionality. Upon presentation of an input pattern the VT and the CT conjointly filter out categories from the selection competition. Only those categories are disqualified, for which the presented pattern does not seem to match the categories' characteristics. It is evident, that for a category j and an input pattern $\mathbf{x}$, if $\mathbf{x}\notin V(\mathbf{w}_j|\rho)$ or $\mathbf{x}\notin C(\mathbf{w}_j|a,\omega)$ or both, j will fail the VT, the CT or both respectively and will not have a chance to be selected $\mathbf{x}$. Thus, for j to be potentially selected by $\mathbf{x}$ it must at least hold that $\mathbf{x}\in V(\mathbf{w}_j|\rho)\cap C(\mathbf{w}_j|a,\omega)$. This last observation motivates the next definition:

Definition 4

For the Choice-by-Difference CCF we define as *claim region* $L(\mathbf{w}_j|\rho,a,\omega)$ of an EA category j with template $\mathbf{w}_j$ for particular values of the parameters ρ, a and ω the subset of R^M

$$L(\mathbf{w}_j \mid \rho,a,\omega) = V(\mathbf{w}_j \mid \rho) \cap C(\mathbf{w}_j \mid a,\omega) \Leftrightarrow \begin{cases} L(\mathbf{w}_j \mid \rho,a,\omega) = \left\{\mathbf{x}\in R^M \mid dis(\mathbf{x},\mathbf{w}_j) \le d_L(\mathbf{w}_j \mid \rho,a,\omega)\right\} \\ d_L(\mathbf{w}_j \mid \rho,a,\omega) = \min\left\{d_V(\mathbf{w}_j \mid \rho), d_C(\mathbf{w}_j \mid a,\omega)\right\} \end{cases}. \quad (20)$$

As expected, the quantity $d_L(\mathbf{w}_j|\rho,a,\omega)$ is called the *radius of the claim region*, which also decreases, when a category's size increases. From Figures 4 and 5 we have observed that both the match and the choice region have a similar shape; in general, the regions differ only in their radii. Also, in view of Definition 4 we expect that the claim region will coincide either with the match or the choice region depending on the values of ρ, a and ω.

Property 6

For the Choice-by-Difference CCF the claim region $L(\mathbf{w}_j|\rho,a,\omega)$ of an EA category j with template $\mathbf{w}_j$ coincides either with the category's match region $V(\mathbf{w}_j|\rho)$ or its choice region $C(\mathbf{w}_j|a,\omega)$ depending on the value of the vigilance parameter ρ, the value of the choice parameter a, the value of ω and, under certain circumstances, on the category's size $s(\mathbf{w}_j)$. For $a\in(0,1)$ and $\omega\in[1/2,\infty)$ we discriminate 3 major cases:

i) If $0\le\rho\le\rho^+$, then $L(\mathbf{w}_j|\rho,a,\omega)=C(\mathbf{w}_j|a,\omega)$.

ii) If $\rho^+<\rho<\rho^-$ and we define $s_{thres} = D\left(\frac{1-\rho}{1-a} - 2\omega + 1\right)$, then

- iia) if $s(\mathbf{w}_j)<s_{thres}$, then $L(\mathbf{w}_j|\rho,a,\omega)=C(\mathbf{w}_j|a,\omega)$.
- iib) if $s_{thres}<s(\mathbf{w}_j)$, then $L(\mathbf{w}_j|\rho,a,\omega)=V(\mathbf{w}_j|\rho)$.
- iic) if $s(\mathbf{w}_j)=s_{thres}$, then $L(\mathbf{w}_j|\rho,a,\omega)=C(\mathbf{w}_j|a,\omega)=V(\mathbf{w}_j|\rho)$.

iii) If $\rho^-\le\rho\le1$, then $L(\mathbf{w}_j|\rho,a,\omega)=V(\mathbf{w}_j|\rho)$.

An immediate result stemming from Definition 4, Properties 3 and 5 is the following:

Property 7

For all $\rho\in[0,1]$, $a\in(0,1)$, $\omega\in[1/2,\infty)$ and $\gamma\in(0,1]$ the claim region of any EA category j decreases in terms of hyper-volume each time the category is updated due to an input pattern. That is, if $\mathbf{x}\in L(\mathbf{w}_j^{old}|\rho,a,\omega)\text{-}R(\mathbf{w}_j^{old})$, then $Vol(L(\mathbf{w}_j^{old}|\rho,a,\omega))< Vol(L(\mathbf{w}_j^{new}|\rho,a,\omega))$.

4. RESULTS APPLICABLE TO EA & EAM

The definition of the three category regions along with their properties are useful towards the derivation of some interesting results regarding EA and EAM. All the results of this section apply for EA modules with parameters $\rho\in[0,1]$, $a\in(0,1)$ and $\omega\in[1/2,\infty)$. Result 1 presented below follows immediately from Property 6. Depending on the values of ρ, a,ω and occasionally from a category's size, satisfaction of the VT by a category with respect to an input pattern will automatically imply the simultaneous satisfaction of the CT with respect to the same pattern and vice versa. Again, note that all the results presented in this section are valid for any value $\mu\in(0,1]$.

Result 1
For the Choice-by-Difference CCF, during the training or performance phase of an EA/EAM network, for all categories and all input patterns satisfaction of the VT implies simultaneous satisfaction of the CT and vice versa depending on the value of the network parameters ρ, a, ω and, under certain circumstances, on the category's size. For all input patterns and existing categories:

i) If If $0 \le \rho \le \rho^+$, then it suffices to perform only the CT.

ii) If $\rho^+ < \rho < \rho^-$ and we define $s_{thres} = D\left(\frac{1-\rho}{1-a} - 2\omega + 1\right)$, then

- iia) if $s(\mathbf{w}_j) < s_{thres}$, then it suffices to perform only the CT.
- iib) if $s_{thres} < s(\mathbf{w}_j)$, then it suffices to perform only the VT.
- iic) if $s(\mathbf{w}_j) = s_{thres}$, then perform either the CT or the VT.

iii) If $\rho^- \le \rho \le 1$, then it suffices to perform only the VT.

Definition 5
We define as *ρ-insensitive parameter domain* of the (a,ρ) parameter space the subset of $(0,1)\times[0,1]$ for which $0 \le \rho \le \rho^+$ and as *ω-insensitive parameter domain* of the (a,ρ) parameter space the subset of $(0,1)\times[0,1]$ for which $\rho^- \le \rho \le 1$.

Based on this definition and Result 1 we reach the following conclusion:

Result 2
If an EA/EAM network uses Choice-by-Difference CCF and operates in the ρ-insensitive (or ω-insensitive) parameter domain of the (a,ρ) parameter space, then its training and performance phase does not depend on the specific value of ρ (or ω).

Result 2 tells us, for example, that, if an EA network operates in the ρ-insensitive parameter domain, the number of categories it is going to create during training does not depend on the specific choice of ρ as long as $0 \le \rho \le \rho^+$. A similar statement can be made if the network is operating in the ω-insensitive parameter domain. An immediate result derived from Properties 1, 4 and 6 is the following statement pertaining to the maximum size of categories.

Result 3
If an EA/EAM network uses Choice-by-Difference CCF and has been trained with a finite cardinality training set, the size of EA categories obeys the following restrictions

i) If $0 \le \rho \le \rho^+$, then for any category j it will hold that $s(\mathbf{w}_j) < D(1-\rho^+)$.

ii) If $\rho^- \le \rho \le 1$, then for any category j it will hold that $s(\mathbf{w}_j) \le D(1-\rho)$.

Both statements can be combined in a single inequality

$$s(\mathbf{w}_j) \le D\left[1 - \max\{\rho, \rho^+\}\right]. \tag{21}$$

Figure 6 illustrates different regions in the (a,ρ) parameter space for different values of ω. Region **I** is the ρ-insensitive parameter domain, where satisfaction of the CT implies automatic satisfaction of the VT and $s(\mathbf{w}_j) < D(1-\rho^+)$ for every j. The "mixed" region **II** corresponds to all pairs (a,ρ) that satisfy the condition $\rho^+ < \rho < \rho^-$, where the necessity of VT or CT depends on the size of the category. Finally region **III** is the ω-insensitive parameter domain, where satisfaction of the VT implies automatic satisfaction of the CT and $s(\mathbf{w}_j) \le D(1-\rho)$ for every j. The last region can be further refined into 2 sub-regions $\mathbf{III}_a$ and $\mathbf{III}_b$. The first (or second) one stands for all choices of (a,ρ) in the ω-insensitive parameter domain, for which uncommitted nodes are (or are not) competitive, that is, $T_u > T_{min}$ (or $T_u \le T_{min}$). We observe that, as $\omega \to \infty$, the ω-insensitive parameter domain finally dominates. This is to be expected, since for $\omega \to \infty$ we have that $T_u \to -\infty$ and uncommitted nodes are not competitive anymore.

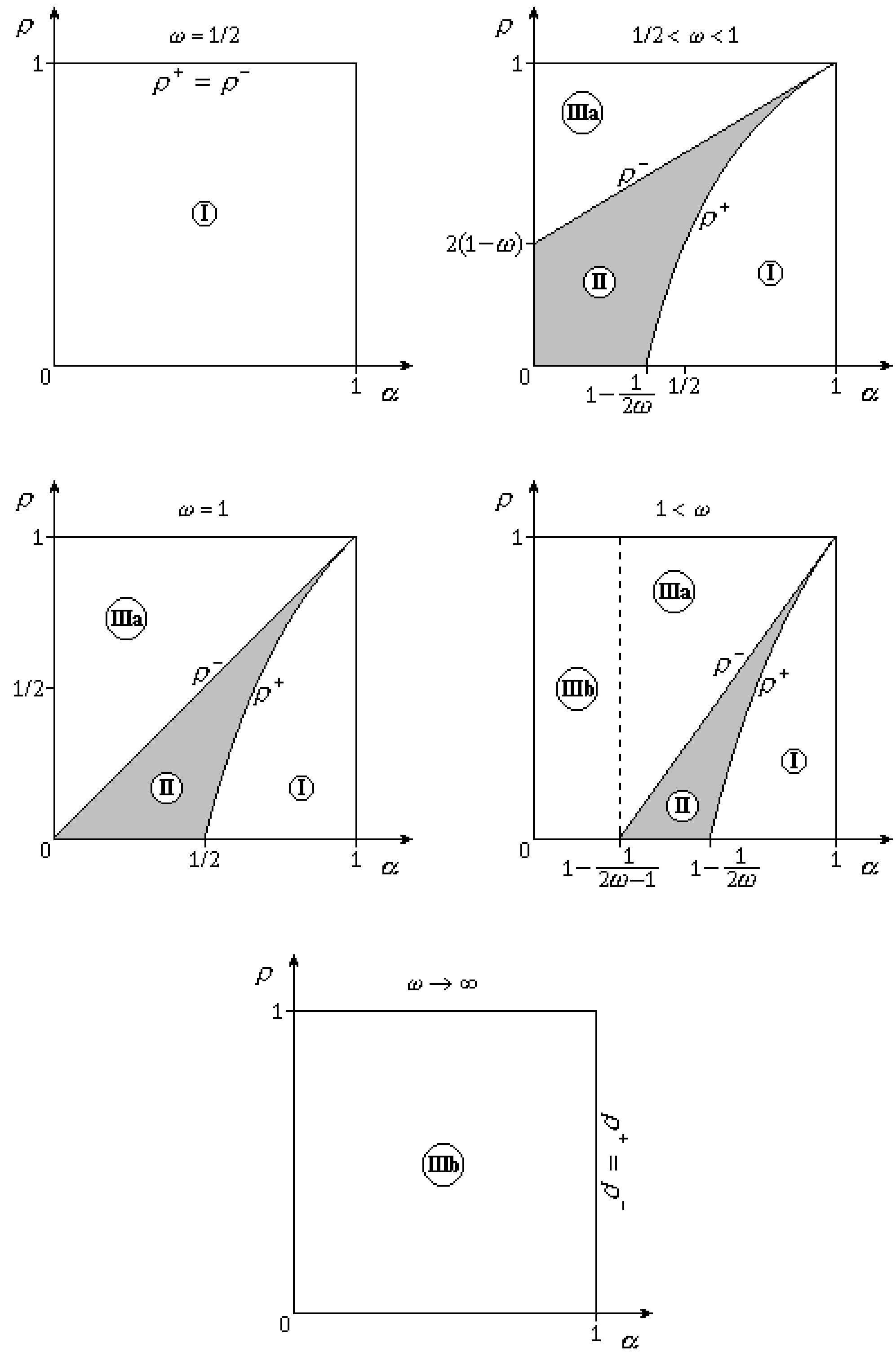

Figure 6: Regions of interest in the (a,ρ) parameter domain for successively larger values of ω.

As a final note it is worth mentioning that, due to the similar design of EA/EAM's and FA/FAM, all the results that we have presented in this paper can be easily extended to FA/FAM with some small modifications: μ can be ignored, since there is no FA/FAM counterpart parameter, D must be replaced by M and ω must be replaced with $w_u \geq 1$ (the quantity, with which the weights of uncommitted nodes is being initialized prior to network operation).

5. CONCLUSIONS

In this paper we have presented the concept of Ellipsoid ART (EA) category regions and their associated properties as they develop via the use of the Choice-by-Difference (CBD) category choice function. Based on their properties we have established the similarity in role of the vigilance test (VT) and the participation of uncommitted F_2-layer nodes in the competition process (commitment test – CT) within an EA module. We have shown that upon presentation of an input pattern the two tests conjointly filter out categories, for which the presented pattern does not match their characteristics. Moreover, based again on the category regions' properties, we have presented 3 theoretical results pertaining to both the training and performance phase of Ellipsoid ART and the Ellipsoid ARTMAP classifier. In specific, we have shown that, depending on the value of their network parameters, for all existing categories in an EA module and all presented input patterns satisfaction of the VT implies simultaneous satisfaction of the CT and vice versa. Under the same setting, the maximum size of EA categories that can be constructed during training also depends on the values of the network parameters. Finally, we note that all the results presented here can be applied to Fuzzy ART and Fuzzy ARTMAP with some small modifications.

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