Packet Error Probabilities in Frequency-Hopped Spread-Spectrum Packet Radio Networks—Memoryless Frequency-Hopping Patterns Considered

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Abstract—**In this paper, we compute the packet error probability induced in a frequency-hopped spread-spectrum packet radio network. The frequency spectrum is divided into q frequency bins. Each packet is exactly one codeword from an (M, L) Reed-Solomon code [M = number of codeword symbols (bytes); L = number of information symbols (bytes)]. Every user in the network sends each of the M bytes of his packet at a frequency chosen among the q frequencies with equal probability, and independently of the frequencies chosen for other bytes (i.e., memoryless frequency-hopping patterns). Furthermore, statistically independent frequency-hopping patterns correspond to different users in the network. Provided that K users have simultaneously transmitted their packets on the channel and a receiver has locked on to one of these K packets, we evaluate the probability that this packet is not decoded correctly. We also show that, although memoryless frequency-hopping patterns are utilized, the byte errors at the receiver are not statistically independent; instead they exhibit a Markovian structure.**

I. Introduction

THE rapid growth of computer communication has motivated an intense interest in packet switching radio techniques [1]. Furthermore, there is a growing need for computer communication and information distribution in tactical military applications where spread-spectrum waveforms must be used in order to achieve reliable operation in the presence of intentional interference (jamming). As a result, a thorough investigation of spread-spectrum packet radio networks becomes necessary.

The bit error probability induced in frequency-hopped spread-spectrum systems has been examined before [2]. In spread-spectrum packet radio networks, the computation of the packet error probability is more important than the bit error probability. Various authors [3], [4], [5] have examined the packet error probability in a frequency-hopped spread-spectrum system, which utilizes memoryless frequency-hopping patterns.

In [3] and [5], a packet is divided into M bytes and the frequency spectrum consists of q frequency bins. Each byte of a packet is transmitted at a frequency chosen from the q frequencies with equal probability, independently of the frequencies chosen for other bytes (memoryless hopping patterns). Furthermore, different packets have statistically independent frequency-hopping patterns. In [3] and [5], it is stated that if K packet transmissions occur simultaneously over the channel and a receiver locks on to one of these packets, then the byte errors of this packet, given K, are independent. As we show in Section III, this statement is true only for $K = 2$; for values of K greater than 2, the byte errors exhibit a Markovian structure. Then, in the same section, we describe a method to compute the packet error probability $P_e(K)$. Finally, we present some numerical results ($P_e(K)$ versus K) when Reed–Solomon codes are used for the encoding of packets.

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II. The Model

In our model of a frequency-hop packet radio network, each user employs a random frequency-hopping pattern. Every frequency-hopping pattern is a sequence of independent random variables, each of which is uniformly distributed over a set of q frequencies (memoryless frequency-hopping patterns). Different users in the network utilize statistically independent frequency-hopping patterns. The dwell interval and the hop interval are assumed to be the same. Furthermore, we assume the following:

1) a packet is *exactly one codeword* from an (M, L) Reed–Solomon code (RS) [M = total number of codeword symbols (bytes); L = number of information symbols (bytes)],

2) each dwell (hop) interval contains *only one codeword symbol (byte)* of the Reed-Solomon code.

The channel time is divided into slots and the users in the network initiate their packet transmissions at the beginning of some slot. Suppose that $K (K \geq 2)$ packets (i.e., packet 1, packet 2, $\cdots$, packet K) are transmitted in a slot, and a receiver locks on to packet 1. We say that a byte of packet 1 is hit if, during its reception by the receiver, at least one of the other packets (i.e., packets 2, 3, $\cdots$, K) occupies the same frequency bin that packet 1 occupies. Let us denote by $P_e(K)$ the probability that packet 1 is incorrectly decoded, given that the RS code corrects at most e byte errors and $K - 1$ other packets are transmitted in the same slot. In the next section, we will present a method to compute $P_e(K)$, under the following assumptions:

3) the only noise present is due to multiple access interference,

4) a byte hit results in a byte error, and this is an upper bound for models where a hit results in an increased symbol (byte) error probability.

It is worth noting that assumption 4 is also adopted in [3] and [4].

III. The Markovian Structure of Byte Errors—A Method to Compute the Packet Error Probability

We assume that $K (K \geq 2)$ packets (i.e., packet 1, packet 2, $\cdots$, packet K) are transmitted in the same slot and a receiver locks on to packet 1. These packets correspond to K different users in the network. We denote by $[f_j^{(i)}]_{j=1}^{M}$ the frequency-hopping pattern corresponding to user i. In Fig. 1, we show a

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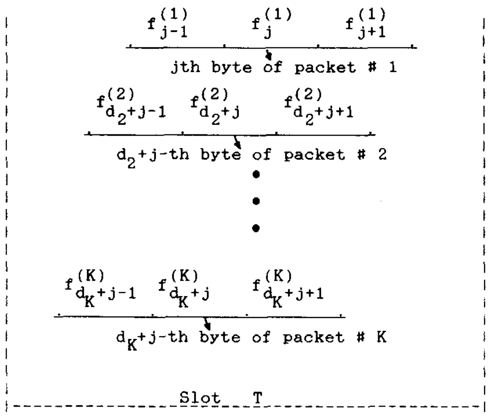

Fig. 1. A realization of K packet arrivals at the receiver site.

realization of the K packets at the receiver site. The d_i's ($2 \le i \le K$) in Fig. 1 depend on the difference between the arrival times of packet i and packet 1 (i.e., the packet on which the receiver is locked) at the receiver; the d_i's ($2 \le i \le K$) take integer values.

Let us now denote by H_j, ($1 \le j \le M$), random variables such that $H_j = 1$ if the jth byte of packet 1 is hit and $H_j = 0$ otherwise. We observe from Fig. 1 that H_j depends on $f_j^{(1)}$, $f_{d_2+j}^{(2)}, f_{d_2+j+1}^{(2)}, \cdots, f_{d_K+j}^{(K)}, f_{d_K+j+1}^{(K)}$. Hence, we conclude that H_j ($1 \le j \le M$) is independent of $H_{j-2}, H_{j-3}, \cdots H_1$, but depends on H_{j-1}. Furthermore, in Appendix A we show that

$$\Pr(H_j=0)=(1-q^{-1})^{2(K-1)}; \qquad 1\le j\le M. \tag{1}$$

$$\Pr(H_j=1)=1-(1-q^{-1})^{2(K-1)}; \qquad 1\le j\le M. \tag{2}$$

$$\begin{aligned}\Pr(H_j=0 \cap H_{j-1}=0)=(1-q^{-1})^{2(K-1)} \cdot [q^{-1} \\ \cdot (1-q^{-1})^{K-1}+(1-q^{-1}) \\ \cdot (1-2q^{-1})^{K-1}]; \qquad 2\le j\le M.\end{aligned} \tag{3}$$

$$\begin{aligned}\Pr(H_j=0 \cap H_{j-1}=1)=(1-q^{-1})^{2(K-1)} \cdot [1-q^{-1} \\ \cdot (1-q^{-1})^{K-1}-(1-q^{-1}) \\ \cdot (1-2q^{-1})^{K-1}]; \qquad 2\le j\le M\end{aligned} \tag{4}$$

$$\Pr(H_j=1 \cap H_{j-1}=0)=\Pr(H_j=0 \cap H_{j-1}=1); \qquad 2\le j\le M \tag{5}$$

and

$$\begin{aligned}\Pr(H_j=1 \cap H_{j-1}=1)=1-(1-q^{-1})^{2(K-1)} \cdot [2-q^{-1} \\ \cdot (1-q^{-1})^{K-1}-(1-q^{-1}) \\ \cdot (1-2q^{-1})^{K-1}]; \qquad 2\le j\le M.\end{aligned} \tag{6}$$

Formulas (1)–(6) show that "byte hits" are independent only when $K = 2$; for other values of K, "byte hits" exhibit a Markovian structure. The same structure is also exhibited by byte errors, since whether a byte is received in error or not depends solely on whether this byte is hit or not (see model in Section II).

We will now compute the probability $P_e(K)$ ($P_e(K)$ was defined in Section II). We denote by $S(m, n)$, $1 \le n \le M$, $m \le n$, the number of bytes from byte m to byte n of packet 1 which are in error. We also define

$$p(i)=\Pr(H_j=i); \qquad i=0,\,1,\ 1\le j\le M \tag{7}$$

and

$$p(i/k)=\Pr(H_j=i/H_{j-1}=k); \qquad i,\,k=0,\,1,\ 2\le j\le M. \tag{8}$$

Then, it is easy to verify the following expressions (formulas (10)–(13) are verified in Appendix B).

$$P_e(K)=\sum_{i=e+1}^{M}\Pr[S(1, M)=i]. \tag{9}$$

$$\Pr(S(1, 1)=i)=p(i); \qquad i=0,\,1. \tag{10a}$$

$$\begin{aligned}\Pr(S(1, n)=i)=\Pr(S(2, n)=i/H_1=0)p(0) \\ +\Pr(S(2, n)=i-1/H_1=1)p(1); \\ 2\le n\le M,\ 0\le i\le n.\end{aligned} \tag{10b}$$

$$\Pr(S(2, 2)=i/H_1=k)=p(i/k); \qquad i,\,k=0,\,1. \tag{11}$$

$$\begin{aligned}&\Pr(S(2, n)=i/H_1=0) \\ &=\Pr(S(2, n-1)=i/H_1=0)\cdot p(0/0) \\ &\quad+\Pr(S(2, n-1)=i-1/H_1=1)\cdot p(1/0); \\ &\quad 3\le n\le M,\ 0\le i\le n-1.\end{aligned} \tag{12}$$

$$\begin{aligned}&\Pr(S(2, n)=i/H_1=1) \\ &=\Pr(S(2, n-1)=i/H_1=0)\cdot p(0/1) \\ &\quad+\Pr(S(2, n-1)=i-1/H_1=1)\cdot p(1/1); \\ &\quad 3\le n\le M,\ 0\le i\le n-1.\end{aligned} \tag{13}$$

In formulas (10b), (12), and (13) we make the convention that $\Pr(S(m, n) = i/H_1 = k) = 0$ if $i < 0$ or $i > n - m$ for every n, m, k such that $1 \le n \le M$, $m \le n$, and $k = 0$ or 1. We can easily compute $P_e(K)$ based on expressions (10)–(13). We start from $n = 3$, and we evaluate the probabilities $\Pr(S(2, n) = i/H_1 = 0)$ and $\Pr(S(2, n) = i/H_1 = 1)$ for $i = 0, 1, \cdots, n - 1$ based on expressions (11)–(13); then we perform similar computations for $n = 4, 5, \cdots$ up to $n = M$. Finally, we end up having computed $\Pr(S(2, M) = i/H_1 = 0)$ and $\Pr(S(2, M) = i/H_1 = 1)$ for $i = e + 1, e + 2, \cdots M$. So, we can evaluate the probabilities $\Pr(S(1, M) = i)$ for $i = e + 1, e + 2, \cdots M$ through expression (10b). As a result, we can find $P_e(K)$ via formula (9).

For the numerical results both the (31, 15) and the (127, 63) Reed–Solomon (RS) codes are considered. In Table I, we have tabulated $P_e(K)$ versus K for the (31, 15) and the (127, 63) RS codes, when $q = 50$ and $q = 100$.

IV. Comments and Conclusions

The consideration of a slotted channel (see model in Section II) is not so restrictive. The packet error probability, $P_e(K)$, computed in the previous section for the slotted channel is an

TABLE I
EXACT PACKET ERROR PROBABILITIES

RS-(31,15), e=8, q=50

K	.... $P_e(K)$.....
2	0.21799508D-05
3	0.42633287D-03
4	0.63364012D-02
5	0.33138681D-01
6	0.98737437D-01
7	0.20804580
8	0.34838999
9	0.49774973
10	0.63577182

RS-(127,63), e=32, q=50

K	....$P_e(K)$.....
3	0.49235D-09
4	0.43842440D-05
5	0.83349435D-03
6	0.19909335D-01
7	0.13361788
8	0.39520434
9	0.69143285
10	0.88569988

RS-(31,15), e=8, q=100

K	....$P_e(K)$.....
2	0.663500D-08
3	0.20954807D-05
4	0.49792305D-04
5	0.41111387D-03
6	0.19052785D-02
7	0.61372921D-02
8	0.15402369D-01
9	0.32242764D-01
10	0.58841087D-01
11	0.96487248D-01
12	0.14528345
13	0.20413414
14	0.27097077
15	0.34311226
16	0.41765775
17	0.49183217
18	0.56323945
19	0.63000987
20	0.69085063

RS-(127,63), e=32, q=100

K	....$P_e(K)$.....
5	0.43445D-09
6	0.8088586D-07
7	0.39495779D-05
8	0.76887814D-04
9	0.76684915D-03
10	0.46105686D-02
11	0.18693216D-01
12	0.55431988D-01
13	0.12783094
14	0.24057638
15	0.38429430
16	0.53850531
17	0.68097367
18	0.79641081
19	0.87971080
20	0.93393379

TABLE II
COMPARISON OF EXACT AND APPROXIMATE PACKET ERROR PROBABILITIES

RS-(31,15), e=8, q=50

K	....$P_e(K)$.....	$\lvert P_e(K)-\hat{P}_e(K)\rvert / P_e(K)\times 100\%$
2	0.21799508D-05	0.0%
3	0.42558507D-03	0.2%
4	0.63257070D-02	0.2%
5	0.33095979D-01	0.1%
6	0.98653594D-01	0.08%
7	0.20794990	0.05%
8	0.34833579	0.02%
9	0.49777976	0.006%
10	0.63589453	0.02%

RS-(127,63), e=32, q=50

K	....$P_e(K)$.....	$\lvert P_e(K)-\hat{P}_e(K)\rvert / P_e(K)\times 100\%$
3	0.48950D-09	0.6%
4	0.43615241D-05	0.5%
5	0.83050917D-03	0.4%
6	0.19869135D-01	0.2%
7	0.13350705	0.08%
8	0.39516112	0.01%
9	0.69155920	0.02%
10	0.88587198	0.02%

upper bound on the packet error probability induced in the unslotted channel, provided that $K - 1$ corresponds to the maximum number of interfering packets during the reception of packet 1 (for more details see [4, Sect. IV-B]. Furthermore, the pessimistic assumption (assumption 4) in Section II) that a byte hit results in a byte error need not be made either. More optimistic assumptions described in [5, Sect. IV] where thermal noise is also present can be incorporated in our model, too. They will simply make the presentation of Section III more complicated.

In Table II, we have computed the packet error probability $\hat{P}_e(K)$ induced by our system (see the model in Section II) if the byte errors are treated as independent for all values of K. In doing so, we took the byte error probability equal to $p(1)$ [$p(1)$ is defined in formulas (2) and (7) of Section III]. Once more the (31, 15) and (127, 63) RS codes are considered, and q is taken equal to 50. Tables I and II show that there is good agreement between $P_e(K)$ and $\hat{P}_e(K)$, at least for the examples considered.

Concluding, we remark that in this paper, we have described a method of computing the packet error probability $P_e(K)$ induced in the spread-spectrum system of Section II, which utilizes memoryless frequency-hopping patterns. We have also shown (in Section III) that a very simple computer program can be written to compute $P_e(K)$ via formulas (10)–(13).

APPENDIX A

A. Sketchy Proof of Formulas (1) and (2)

$$\begin{aligned}
\Pr(H_j=0) &= \Pr(\text{no hit}) \\
&= \sum_{t=1}^{q} \Pr(\text{no hit}/f_j^{(1)}=t) \\
&\quad \cdot \Pr(f_j^{(1)}=t) \text{ [see Fig. 1]} \\
&= \sum_{t=1}^{q} (1-q^{-1})^{2(K-1)} \cdot q^{-1} \\
&= (1-q^{-1})^{2(K-1)}. \qquad \text{(A.1)}
\end{aligned}$$

From (A.1) we get that

$$\Pr(H_j=1) = 1-(1-q^{-1})^{2(K-1)} \qquad \text{(A.2)}$$

B. Sketchy Proof of Formulas (3), (4), (5), and (6)

$$\begin{aligned}
&\Pr(H_j=0 \cap H_{j-1}=0) \\
&\quad = \Pr(\text{no hit}/f_j^{(1)}=f_{j-1}^{(1)}) \cdot \Pr(f_j^{(1)}=f_{j-1}^{(1)}) \\
&\quad\quad + \Pr(\text{no hit}/f_j^{(1)} \neq f_{j-1}^{(1)}) \\
&\quad\quad \cdot \Pr(f_j^{(1)} \neq f_{j-1}^{(1)}) \text{ [see Fig. 1]} \\
&\quad = (1-q^{-1})^{3(K-1)} \cdot q^{-1} + (1-q^{-1})^{2(K-1)} \\
&\quad\quad \cdot (1-2q^{-1})^{K-1} \cdot (1-q^{-1}). \qquad \text{(A.3)}
\end{aligned}$$

Formulas (4), (5), and (6) can be verified if we observe that

$$\begin{aligned}
\Pr(H_j=0 \cap H_{j-1}=1) &= \Pr(H_j=0) \\
&\quad - \Pr(H_j=0 \cap H_{j-1}=0). \qquad \text{(A.4)}
\end{aligned}$$

$$\begin{aligned}
\Pr(H_j=1 \cap H_{j-1}=0) &= \Pr(H_{j-1}=0) \\
&\quad - \Pr(H_j=0 \cap H_{j-1}=0). \qquad \text{(A.5)}
\end{aligned}$$

and

$$\begin{aligned}
\Pr(H_j=1 \cap H_{j-1}=1) &= 1-2 \cdot \Pr(H_j=0 \cap H_{j-1}=1) \\
&\quad - \Pr(H_j=0 \cap H_{j-1}=0). \qquad \text{(A.6)}
\end{aligned}$$

Hence, using (A.1), (A.3), and (A.4)–(A.6), we get (4)–(6).

APPENDIX B

Formula (10a) is obvious from the definition of $S(1, 1)$.

A. Proof of Formula (10b)

$$\begin{aligned}\Pr(S(1, n)=i) &= \Pr(S(1, n)=i/H_1=0) \\ &\quad \cdot \Pr(H_1=0)+\Pr(S(1, n)=i/H_1=1) \\ &\quad \cdot \Pr(H_1=1)=\Pr(S(2, n)=i/H_1=0) \\ &\quad \cdot p(0)+\Pr(S(2, n)=i-1/H_1=1) \\ &\quad \cdot p(1). \end{aligned} \tag{B.1}$$

(B.1) is formula (10b); note that the second term of the right-hand side (RHS) of (B.1) is zero if $i = 0$, while if $i = n$ the first term of the RHS of (B.1) is zero.

Formula (11) is also a direct consequence of the definition of $S(2, 2)$.

B. Proof of Formula (12)

$$\begin{aligned}&\Pr(S(2, n)=i/H_1=0) \\ &\quad =\Pr(S(2, n)=i/H_1=0, H_2=0) \\ &\quad \cdot \Pr(H_2=0/H_1=0)+\Pr(S(2, n)=i/H_1=0, H_2=1) \\ &\quad \cdot \Pr(H_2=1/H_1=0). \end{aligned} \tag{B.2}$$

From (B.2), due to the Markovian structure of byte errors (see [6, formula (3), p. 5] for more details), we take

$$\begin{aligned}&\Pr(S(2, n)=i/H_1=0) \\ &\quad =\Pr(S(2, n)=i/H_2=0) \cdot p(0/0) \\ &\qquad +\Pr(S(2, n)=i/H_2=1) \cdot p(1/0) \\ &\quad =\Pr(S(3, n)=i/H_2=0) \cdot p(0/0) \\ &\qquad +\Pr(S(3, n)=i-1/H_2=1) \cdot p(1/0). \end{aligned} \tag{B.3}$$

From (B.3) and the stationarity of the Markov chain under consideration [see formulas (1)–(6)], we conclude that

$$\begin{aligned}&\Pr(S(2, n)=i/H_1=0) \\ &\quad =\Pr(S(2, n-1)=i/H_1=0) \cdot p(0/0) \\ &\qquad +\Pr(S(2, n-1)=i-1/H_1=1) \cdot p(1/0). \end{aligned} \tag{B.4}$$

(B.4) is formula (12); once more, note that if $i = 0$ the second term of the RHS of (B.4) is zero, while if $i = n - 1$ the first term of the RHS of (B.4) is zero.

Formula (13) can be proven following arguments similar to those used for the proof of formula (12).

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