

# ON THE ERROR PROBABILITY OF FREQUENCY-HOPPED SPREAD-SPECTRUM MULTIPLE-ACCESS SYSTEMS WITH MORE THAN ONE CODE SYMBOLS PER DWELL INTERVAL

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## ABSTRACT

This paper is concerned with the multiple access capability of an asynchronous, frequency-hopped spread-spectrum communication system employing error correcting codes. Many current frequency hopped spread spectrum systems employ error correcting codes with more than one code symbol per dwell interval. In this paper we present a method to compute the codeword error probability induced in such spread spectrum systems. Furthermore, comparisons with spread spectrum systems utilizing one code symbol per dwell interval are conducted, and useful conclusions are drawn.

## 1 Introduction

Many current frequency hopped spread spectrum systems employ error correcting codes with more than one code symbol per dwell interval. This paper examines the multiple access capability of these systems. In particular, we present a method to compute the induced codeword error probability. Furthermore, some useful conclusions are drawn by comparing the codeword error probabilities induced in spread spectrum systems utilizing more than one versus one (see [1],[2],[3]) code symbols per dwell interval.

## 2 The Model

We adopt the model in [4]. The only difference between our model and the model in [4] is that our spread spectrum system employs  $s(s > 1)$  instead of one ( $s = 1$ ) code symbols per dwell interval. The  $i$ th ( $1 \leq i \leq s$ ) symbol of a dwell interval is called

symbol  $i$ . We assume, as in [4], that a packet consists of only one codeword. Hence, we can use the words packet and codeword interchangeably.

We introduce some notation with the aid of Figure 1, where  $s = 2$ . Each user employs a random frequency hopping pattern with frequencies chosen uniformly from the set  $Q = \{1, 2, \dots, q\}$  and independently of the frequencies chosen by the other users. We denote the frequency hopping pattern for user  $k$  as  $\{F_j^k; j = \dots, -2, -1, 0, 1, \dots\}$ . Suppose that  $K$  users are transmitting packets and a receiver locks on to the packet of user 1. We assume that user 1's packet consists of symbols transmitted using frequencies  $F_1^1, F_2^1, \dots, F_N^1$ , where  $N$  corresponds to the the number of dwell intervals per packet. It is worth noting that  $N$  is equal to  $\lceil M/s \rceil$ , where  $M$  is the number of code symbols per packet. We assign indices to the  $K - 1$  interfering packets (i.e., user 1, user 2, ..., user  $K$ ). Index  $k$  ( $k \in \{2, \dots, K\}$ ) belongs to the index set  $J_i, 1 \leq i \leq s$ , if user  $k$  changes carrier frequency during the reception of symbol  $i$  of user 1's packet by the receiver. Obviously,  $\bigcup_{i=1}^s J_i = \{2, \dots, K\}$ . Two dwell intervals of user  $k$  ( $k \in \{2, \dots, K\}$ ) overlap with the  $j$ th dwell interval of user 1 (see also Figure 1). We define the frequency utilized by the dwell interval on the left as  $F_j^k$ . We also define the collection of frequencies  $F_j^k, k \in J_i$ , as  $\tilde{F}_{j,i}$ , and the collection of frequencies  $\tilde{F}_{j,i}, 1 \leq i \leq s$ , as  $\tilde{F}_j$ .

## 3 A Method to compute the packet error probability

Suppose that  $K$  users are transmitting packets and a receiver locks on to the packet of user 1. Let us also assume that the cardinality of the set  $J_i (1 \leq i \leq s)$  is equal to  $K_i$ . We define by  $P_e(K_1, \dots, K_s)$  the probability that user 1's packet (i.e., the desired transmission) is decoded incorrectly by the receiver. Let us denote by  $S_l^n, 1 \leq n \leq N; l \leq n$ , the number of code symbols of user 1's packet, from dwell interval

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$l$  to dwell interval  $n$ , that are in error. Then,

$$P_e(K_1, \dots, K_s) = \sum_{m=e+1}^M Pr[S_1^N = m] \quad (1)$$

where  $e$  is the error correction capability of the code. Let us define random vectors  $T_j$ , their realizations  $t_j$ , and binary valued functions  $g_i(x, y_i)$  as follows:

$$T_j = (F_j^1 \tilde{F}_j^1 F_{j+1}^1 \tilde{F}_{j+1}^1) \quad t_j = (f_j^1 \tilde{f}_j^1 f_{j+1}^1 \tilde{f}_{j+1}^1) \\ ; j = 1, 2, \dots \\ g_i(x, y_i) = \begin{cases} 1 & \text{if } x \in y_i \\ 0 & \text{otherwise} \end{cases} \quad (1 \leq i \leq s)$$

where  $x$  takes values from the set  $Q$  ( $Q = \{1, 2, \dots, q\}$ ) and  $y_i$  is a collection of  $K_i$  variables, each one of which assumes values from the set  $Q$ . For example,  $g_1(F_j^1, \tilde{F}_{j,1}^1)$  is a random variable whose value indicates whether the  $j$ th dwell interval of user 1's packet is hit from the left by some user with index in the set  $J_1$ , or not hit from the left by any user with index in the index set  $J_1$  (see also Figure 1).

Let us use the symbols  $\oplus$  and  $\sum$  to denote the sum of two and the sum of an arbitrary number of binary (0,1) variables, respectively. We define the sum of an arbitrary number of binary variables to be equal to one if at least one of the variables is equal to one, and zero otherwise. We finally denote by  $H_i^l, 1 \leq i \leq s; 1 \leq l \leq N$ , binary random variables such that  $H_i^l$  is equal to one if the  $l$ th symbol in the  $l$ th dwell interval of user 1's packet is hit, and zero otherwise.

Our objective is to describe a method which enables us to compute the probabilities  $Pr[S_1^N = m]$  in (1). We first present a useful Lemma.

**Lemma 1** For every  $n \geq 2$ , for all  $m$  such that  $0 \leq m \leq s(n-1)$ , and all  $t_1 \in Q^{2K}$ , the conditional probability  $Pr[S_2^n = m | T_1 = t_1]$  depends on  $t_1$  through the values of the functions  $g_i(f_2^1, \tilde{f}_{2,i}^1), 1 \leq i \leq s$ .

**Proof** We assume that  $t_1$  is such that  $g(f_2^1, \tilde{f}_{2,i}^1) = v_i (v_i = 0, 1), 1 \leq i \leq s$ . We can write

$$Pr[S_2^n = m | T_1 = t_1] = Pr[\sum_{i=1}^s H_i^2 + \sum_{l=3}^n \sum_{i=1}^s H_i^l = m]$$

$$|T_1 = t_1] = Pr[\sum_{i=1}^s \{ \sum_{\sigma=i}^s g_\sigma(F_2^1, \tilde{F}_{2,\sigma}^1) \oplus \sum_{\sigma=1}^i g_\sigma(F_2^1, \tilde{F}_{3,\sigma}^1) \} +$$

$$\sum_{l=3}^n \sum_{i=1}^s H_i^l = m | F_1^1 = f_1^1, \tilde{F}_1^1 = \tilde{f}_1^1, F_2^1 = f_2^1, \tilde{F}_2^1 = \tilde{f}_2^1] =$$

$$Pr[\sum_{i=1}^s \{ \sum_{\sigma=i}^s g_\sigma(f_2^1, \tilde{f}_{2,\sigma}^1) \oplus \sum_{\sigma=1}^i g_\sigma(f_2^1, \tilde{f}_{3,\sigma}^1) \} + \sum_{l=3}^n \sum_{i=1}^s H_i^l =$$

$$m] = Pr[\sum_{i=1}^s \{ \sum_{\sigma=i}^s v_\sigma \oplus \sum_{\sigma=1}^i g_\sigma(f_2^1, \tilde{f}_{3,\sigma}^1) \} + \sum_{l=3}^n \sum_{i=1}^s H_i^l = m] \quad (2)$$

Now because of the assumption that each of the frequencies utilized in a dwell interval were chosen uniformly over  $Q = \{1, 2, \dots, q\}$  and independently of the other frequencies, it can be shown that the probability in (2) is independent of  $f_2^1$ . Thus,  $Pr[S_2^n = m | T_1 = t_1]$  depends on  $t_1$  only through the values of the functions  $g_i(1 \leq i \leq s)$  at the points  $(f_2^1, \tilde{f}_{2,i}^1)$ , respectively. One of the ways of showing that the probability in (2) is independent of  $f_2^1$  is induction (i.e., we prove that this statement is true for  $n=2$  and  $m=0,1,2$ , we assume that it is true for  $n-1$  and all possible  $m$  choices and then we prove that it is true for  $n$  and all possible  $m$  choices). The details are omitted due to lack of space.  $\diamond$

Lemma 1 states an almost obvious fact (see also Figure 1). Given  $t_1$  the number of erroneous code symbols of user 1's packet, from dwell interval two and beyond, depend on whether each one of the groups of users with indices in the index sets  $J_i, 1 \leq i \leq s$ , hits or not the second dwell interval of user 1's packet from the left. Due to Lemma 1 we can write

$$Pr[S_2^n = m | T_1 = t_1] = s(n, m; v_1, \dots, v_s) \\ \text{if } g(f_2^1, \tilde{f}_{2,i}^1) = v_i \text{ for } 1 \leq i \leq s$$

The next theorem shows that the conditional probabilities  $s(n, m; v_1, \dots, v_s)$  satisfy certain recursive expressions

**Theorem 1**

The conditional probabilities  $s(n, m; v_1, \dots, v_s)$  ( $n \geq 3, 0 \leq m \leq s(n-1), v_1, \dots, v_s = 0, 1$ ) satisfy the following recursive expressions.

$$s(n, m; v_1, \dots, v_s) = \\ \sum_{j_1, \dots, j_s=0,1} s(n-1, m - \sum_{i=1}^s \{ \sum_{\sigma=i}^s v_\sigma \oplus \sum_{\sigma=1}^i j_\sigma \}; \\ j_1, \dots, j_s) q^{-1} \prod_{i=1}^s a_i(j_i) + \sum_{j_1, k_1, \dots, j_s, k_s=0,1} s(n-1, \\ m - \sum_{i=1}^s \{ \sum_{\sigma=i}^s v_\sigma \oplus \sum_{\sigma=1}^i j_\sigma \}; k_1, \dots, k_s) (1 - q^{-1}) \\ \prod_{i=1}^s c_i(j_i, k_i) \quad (3)$$

with initial conditions

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$$s(2, m; v_1, \dots, v_s) = \sum_{j_1, \dots, j_s=0,1} \delta(m - \sum_{i=1}^s [\sum_{\sigma=i}^s v_\sigma \oplus \sum_{\sigma=1}^i j_\sigma]) \prod_{i=1}^s a_i(j_i), \quad 0 \leq m \leq s \quad (4)$$

where

$$a_i(j_i) = \begin{cases} (1 - q^{-1})^{K_i} & \text{if } j_i = 0 \\ 1 - (1 - q^{-1})^{K_i} & \text{if } j_i = 1 \end{cases} \quad (5)$$

$$c_i(j_i, k_i) = (1 - 2q^{-1})^{K_i} \quad \text{if } j_i = k_i = 0$$

$$c_i(j_i, k_i) = (1 - q^{-1})^{K_i} - (1 - 2q^{-1})^{K_i} \quad (6)$$

if  $j_i = 0, k_i = 1$  or  $j_i = 1, k_i = 0$

$$c_i(j_i, k_i) = 1 - [c_i(0, 0) + c_i(1, 0) + c_i(0, 1)]$$

if  $j_i = k_i = 1$

and  $\delta$  denotes the delta function.  $\diamond$

The proof of Theorem 1 can be found in [7]. The final step in our effort to compute  $Pr[S_1^N = m]$  is to express this probability in terms of the conditional probabilities  $s(n, m; v_1, \dots, v_s)$ , which can be evaluated recursively with the help of formula (3). It can be shown that

$$Pr[S_1^N = m] = \sum_{j_1, k_1, \dots, j_s, k_s=0,1} s(N, m - \sum_{i=1}^s [\sum_{\sigma=i}^s j_\sigma \oplus \sum_{\sigma=1}^i k_\sigma]);$$

$$k_1, \dots, k_s) q^{-1} \prod_{i=1}^s a_i(j_i) a_i(k_i) +$$

$$\sum_{j_1, k_1, m_1, \dots, j_s, k_s, m_s=0,1} s(N, m - \sum_{i=1}^s [\sum_{\sigma=i}^s j_\sigma \oplus \sum_{\sigma=1}^i k_\sigma]);$$

$$m_1, \dots, m_s) (1 - q^{-1}) \prod_{i=1}^s a_i(j_i) c_i(k_i, m_i) \quad (7)$$

The proof of formula (7) can also be found in [7]. Equations (3)–(7) give us a recursive algorithm, capable of computing  $Pr[S_1^N = m]$  for all possible  $m$  choices. As a result,  $P_e(K_1, \dots, K_s)$  can be readily determined from (1).

## 4 Numerical Results

In an effort to obtain numerical results in a compact form we assume that the arrivals of the interfering users (i.e., users 2, 3, ...,  $K$ ), at the receiver, are uniformly distributed within a dwell interval of user 1's

packet (i.e., the desired transmission). Then, we define the average packet error probability  $P_e(K)$  as follows:

$$P_e(K) = \sum_{K_1 + \dots + K_s = K-1} \frac{(K-1)!}{K_1! \dots K_s!} (s^{-1})^{K-1} P_e(K_1, \dots, K_s)$$

The average packet error probability is chosen to be the measure of performance of our spread spectrum system. It is an indicator of the multiple access capability of the system. In Table 1, we include the values of the average packet error probability induced, when the (32, 16), (64, 32) extended Reed-Solomon codes are used for the encoding of the packets, while  $s = 2$  and  $q = 50$  or 100. Similar results are obtained when  $s=4$  (see [7]).

## 5 Comments-Conclusions

The employment of Reed-Solomon (RS) error correcting codes is justified by the fact that RS codes are most successful in correcting bursts of errors. Error bursts are most frequent in frequency hopped spread spectrum (FH-SS) systems. It is also worth noting, that C. D. Frank et al showed in [5] that RS codes outperform convolutional codes in a frequency hopping system utilizing one code symbol per dwell interval. Furthermore, numerical results quantifying the performance of any block code can be obtained readily, since the analysis, presented in section 3, is valid for any member of the class of block codes.

The assumption that a symbol hit results in a symbol error was made for analytical simplicity. If we were to abolish this assumption, we would have to compute the symbol error probability and take into consideration the interdependence of symbol errors, both of which are difficult tasks, beyond the scope of this work. Note that in this paper we considered only the interdependence of symbol hits. The computational complexity involved in the evaluation of the symbol error probability for a frequency hopped system utilizing BFSK modulation is addressed in [6]. Finally, it is worth mentioning that this assumption leads us to upper bounds on the induced packet error probabilities.

Most researchers in the field ([1],[2],[3]) have concentrated on evaluating codeword error probabilities for frequency hopped spread spectrum systems employing one code symbol per dwell interval. We present these results in Table 2. A comparison of the results in Tables 1 and 2, reveals that RS codes

### 5.4.3.

are most efficient for  $s = 1$  and small  $K$  values or for  $s > 1$  and large  $K$  values. It can be shown that this behavior is exhibited by any member of the class of block codes. Since for most packet error probabilities of interest (i.e., packet error probabilities smaller than  $10^{-2}$ ) the entries in Table 2 are smaller than the entries in Table 1, we conclude that FH-SS systems with one code symbol per dwell interval are more efficient in combatting multiple access interference than FH-SS systems with  $s(s > 1)$  code symbols per dwell interval, when block codes (e.g., RS codes) are used for the encoding of the packets. For FH-SS systems with  $s(s > 1)$  code symbols per dwell interval, it is worth examining whether interleaving the RS codes to degree  $s$  improves the performance of the system.

In this paper, we have computed the average packet(codeword) error probability induced in a frequency hopped spread spectrum system, when more than one code symbols are contained per dwell interval, and when the (32,16),(64,32) or extended RS codes are used for the encoding of the packets. Furthermore, some comparisons with already existing results for FH-SS systems employing one code symbol per dwell interval were conducted, and useful conclusions were drawn.

## References

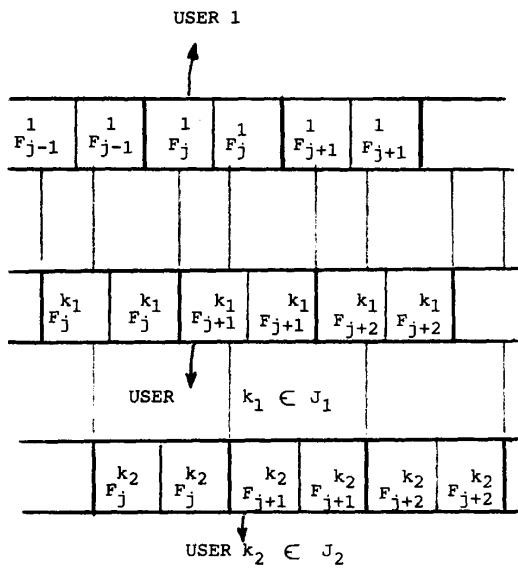
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**Table 1**  
Exact Average Packet Error Probabilities, when  $s=2$

RS-(32,16), $e=8,q=50$		RS-(32,16), $e=8,q=100$	
$K$	$P_e(K)$	$K$	$P_e(K)$
2	0.67800528D-04	2	0.23636125D-05
3	0.11612402D-02	3	0.66687872D-04
4	0.95090789D-02	4	0.44409735D-03
5	0.29878090D-01	5	0.16338776D-02
6	0.67480848D-01	6	0.43371069D-02
7	0.12367458	7	0.93578479D-02
8	0.19636776	8	0.17491890D-01
9	0.28102519	9	0.29428593D-01
10	0.37202369	10	0.45679092D-01
RS-(64,32), $e=16,q=50$		RS-(64,32), $e=16,q=100$	
$K$	$P_e(K)$	$K$	$P_e(K)$
2	0.11299282D-06	3	0.10977481D-06
3	0.37129789D-04	4	0.34243895D-05
4	0.84250091D-03	5	0.36051325D-04
5	0.62876883D-02	6	0.20811685D-03
6	0.25375313D-01	7	0.81902921D-03
7	0.69370159D-01	8	0.24700688D-02
8	0.14520923	9	0.61248515D-02
9	0.25083862	10	0.13071485D-01
10	0.37579355		

**Table 2**  
Packet Error Probabilities, when  $s=1$

RS-(32,16), $e=8,q=50$		RS-(32,16), $e=8,q=100$	
$K$	$P_e(K)$	$K$	$P_e(K)$
2	0.29258205D-05	2	0.90666D-08
3	0.55244200D-03	3	0.28128207D-05
4	0.79367870D-02	4	0.65672672D-04
5	0.40181555D-01	5	0.53292023D-03
6	0.11609753	6	0.24280649D-02
7	0.23771936	7	0.76916175D-02
8	0.38780563	8	0.18989771D-01
9	0.54129833	9	0.39122298D-01
10	0.67758863	10	0.70294109D-01
RS-(64,32), $e=16,q=50$		RS-(64,32), $e=16,q=100$	
$K$	$P_e(K)$	$K$	$P_e(K)$
2	0.3347D-09	3	0.3111D-09
3	0.53409047D-05	4	0.1070926D-06
4	0.65358664D-03	5	0.50014987D-05
5	0.11112115D-01	6	0.78417104D-04
6	0.65402557D-01	7	0.61816346D-03
7	0.20134859	8	0.30402676D-02
8	0.40677557	9	0.10615931D-01
9	0.62152310	10	0.28616755D-01
10	0.79127206		



**Figure 1:** Users 1,  $k_1 \in J_1$ ,  
 $k_2 \in J_2$  at the receiver site

**5.4.5.**