Magnetic Force Calculation Between Thin Coaxial Circular Coils in Air

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We present new and fast procedures for calculating magnetic forces between thin coaxial circular coaxial coils in air. The results are expressed in semianalytical form in terms of the complete elliptical integrals of the first and second kind, Heuman's Lambda function, and a term that must be solved numerically. These expressions are accurate and simple to use for several practical applications. We also describe a comparative method based on the filament technique. We discuss the computational cost and the accuracy of two methods and compare them with already published data. Results obtained by our two approaches are in excellent agreement with each other. They can be used in industrial electromagnetic applications such as electrodynamic levitation systems, linear induction launchers, linear actuators, and coil guns.

***Index Terms*—Filamentary circular coil, magnetic force, thin disk coil, thin wall solenoid.**

I. INTRODUCTION

THE calculation of the magnetic attraction between two current-carrying coils is closely related to the calculation of their mutual inductance. Since their mutual energy is equal to the product of their mutual inductance and the currents in the coils, the component of the magnetic force of attraction or repulsion in any direction is equal to the product of the currents multiplied by the differential coefficient of the mutual inductance taken with respect to that coordinate. As is evident from [1]–[4], the force may be calculated by simple differentiation in cases where a general formula for the mutual inductance is available. We have derived new and improved expressions for magnetic forces of a system composed of two thin wall solenoids, two disk coils (pancakes) and a thin wall solenoid and a thin disk coil (pancake). All coils have the same axe. These accurate expressions are obtained in terms of complete elliptical integrals of the first and second kind, Heuman's Lambda function and a term that must be solved numerically. In this paper, we use Gaussian numerical integration. Even though the mutual inductances of previously mentioned conductors are obtained using two integrations where some integrals cannot be solved analytically or expressed over elliptic integrals [5]–[17], the magnetic forces are obtained semianalytically. Also, we describe another approach based on the filament method [6]–[15] that leads to very simple procedures for calculating the magnetic force between the previously mentioned coils using the well-known formula for Maxwell's coils [1]–[3]. Computed magnetic force values obtained by the two proposed ways are found to be in excellent agreement with each other. The advantages of the proposed procedures compared to numerical methods (FDM, FEM, BEM) are in their simplicity and lower computational cost.

II. BASIC EXPRESSIONS

The magnetic force between two current-carrying coils can be derived from the general expression for their mutual inductance [1]

$$F = I_1 I_2 \frac{\partial M}{\partial z_Q} \tag{1}$$

where I_1 and I_2 are currents of two coils and M is their mutual inductance. z_Q is the generalized coordinate. The magnetic force has only an axial component because the coils are coaxial. We will begin with well-known Maxwell's coils [1]. For these, the mutual inductance is given by

$$M = \mu_0 R_1 R_2 \int_0^{\pi} \frac{\cos\theta \mathrm{d}\theta}{r_Q} \tag{2}$$

where

$$r_Q = \sqrt{z_Q^2 + R_1^2 + R_2^2 - 2R_1R_2\cos\theta}, \quad z_Q = z_2 - z_1.$$

Applying (1), the magnetic force of two coaxial circular filament conductors of radius R_1 and R_2 with corresponding currents I_1 and I_2 as shown in (Fig. 1) is given by

$$F = F_z = \frac{\mu_0 I_1 I_2 k z_Q}{4\sqrt{R_1R_2}} \left[\frac{(2-k^2)}{(1-k^2)}E(k) - 2K(k)\right] \tag{3}$$

where

$$k^2 = \frac{4R_1R_2}{(R_1+R_2)^2 + z_Q^2}.$$

$K(k)$ and $E(k)$ are elliptical integrals of the first and second kind [18].

Expression (3) is fundamental for applying the filament method, which we use as a comparative calculation method in this study.

III. CALCULATION METHOD

A. Two Thin Wall Solenoids

The mutual inductance of two thin wall solenoids shown in Fig. 2 is given by [13] as

$$M = \frac{\mu_0 N_1 N_2 R_1 R_2}{(z_2-z_1)(z_4-z_3)} \int_{z_1}^{z_2}\int_{z_3}^{z_4}\int_0^{\pi} \frac{\cos\theta \mathrm{d}z_I \mathrm{d}z_{II} \mathrm{d}\theta}{r_Q} \tag{4}$$

where

$$r_Q = \sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\theta + (z_I - z_{II})^2}.$$

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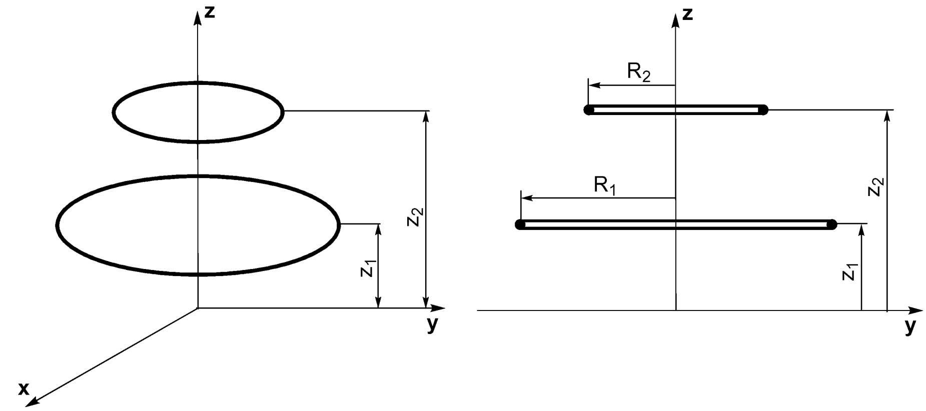

Fig. 1. Maxwell's coils.

The total numbers of turns of the thin wall solenoids are N_1 and N_2. I_1 and I_2 are corresponding currents. Let's introduce the following substitutions:

$$a_1 = z_2 - z_1; \quad a_2 = z_4 - z_3; \quad a_{11} = \frac{a_1 + a_2}{2}$$
$$a_{22} = \frac{a_2 - a_1}{2}; \quad z_{11} = \frac{z_1 + z_2}{2}$$
$$z_{22} = \frac{z_3 + z_4}{2}; \quad z_Q = z_{22} - z_{11}.$$

The mutual inductance (4) can be obtained in an analytical form expressed over elliptical integrals of the first and second kind and the Heuman's Lambda function [13]

$$M = \frac{\mu_0 N_1 N_2 \sqrt{R_1 R_2}}{3 a_1 a_2} \sum_{n=1}^{4} (-1)^{n-1} T_n \tag{5}$$

where

$$T_n = \frac{2(R_2 - R_1)^2 - t_n^2}{k_n} K(k_n) + \frac{t_n^2 - 2R_1^2 - 2R_2^2}{k_n} E(k_n) + \frac{3\pi |t_n|}{4\sqrt{R_1 R_2}} \left[1 - \Lambda_0(\varepsilon_n, k_n)\right].$$

Applying (1), we obtain the magnetic force for the proposed conductor arrangement in an analytical form as

$$F = \frac{\mu_0 N_1 N_2 I_1 I_2}{4 a_1 a_2} \sum_{n=1}^{4} (-1)^{n-1} P_n \tag{6}$$

where

$$\begin{aligned} P_n = & \frac{t_n k_n}{\sqrt{R_1 R_2}} \left[\frac{t_n^2}{k_n'^2} E(k_n) - \left(4R_1R_2 + t_n^2\right) K(k_n) \right] \\ & + \pi sign(t_n) \left|R_2^2 - R_1^2\right| \left[1 - \Lambda_0(\varepsilon_n, k_n)\right] \\ & + \frac{|R_2 - R_1| \left|R_2^2 - R_1^2\right| k_n^3 |t_n| sign(t_n)}{2R_1R_2 k_n'^2 \Delta} \\ & \times \left[E(k_n) - K(k_n) k_n'^2 \sin^2 \varepsilon_n\right] \\ & - \frac{\left|R_2^2 - R_1^2\right| k_n'^2 |t_n| t_n}{2R_1R_2 k_n'^2 \Delta} \left[K(k_n) - E(k_n)\right] \sin\varepsilon_n \cos\varepsilon_n \end{aligned}$$

$$k_n^2 = \frac{4R_1R_2}{(R_1+R_2)^2 + t_n^2}, \quad k_n'^2 = 1 - k_n^2, \quad h = \frac{4R_1R_2}{(R_1+R_2)^2}$$

$$\varepsilon_n = \sin^{-1}\sqrt{\frac{1-h^2}{1-k_n^2}}, \quad \Delta = \sqrt{1 - k_n'^2 \sin^2 \varepsilon_n}$$

$$t_1 = z_Q + a_{11}; \; t_2 = z_Q + a_{22}; \; t_3 = z_Q - a_{11}; \; t_4 = z_Q - a_{22}.$$

K and E are elliptical integrals of the first and second kind and Λ_0 is Heuman's Lambda function [18]. In the previous expressions, the cases $R_1 < R_2$, $R_1 = R_2$ and $R_1 = R_2$ are included. If $z_Q = 0$, then the axial magnetic force is equal zero. This calculation already exists in the literature [1]–[3] but the presented form is more comfortable for numerical applications.

In order to apply the filament method, the thin wall solenoids are then considered to be subdivided into meshes of filamentary coils as shown in Fig. 3. We obtain a set of thin circular coils (Maxwell's coils). Using the same logic as [14] and [15], we obtain the magnetic force of treated system as

$$F = \frac{N_1 N_2}{(2K+1)(2m+1)} \sum_{g=-K}^{g=K} \sum_{l=-m}^{l=m} F(g,l) \tag{7}$$

where

$$F(g,l) = \frac{\mu_0 z(g,l) I_1 I_2 k}{4\sqrt{R_1 R_2}} \left[\frac{2-k^2}{1-k^2} E(k) - 2K(k)\right]$$
$$R_I = R_1, \quad R_{II} = R_2$$
$$z(g,l) = c - \frac{a}{2K+1} g + \frac{b}{2m+1} l$$
$$g = -K, \ldots, 0, \ldots, K; l = -m, \ldots, 0, \ldots, m$$
$$k^2(g,l) = \frac{4R_1R_2}{(R_1+R_2)^2 + z^2(g,l)}$$

N_1 and N_2 are the total number of turns of the thin wall solenoids. I_1 and I_2 are currents that correspond to thin wall solenoids respectively.

B. Two Thin Disk Coils (Pancakes)

The mutual inductance of two thin disk coils (pancakes) shown in Fig. 4 is given by [10] as

$$M = \frac{\mu_0 N_1 N_2}{(R_2 - R_1)(R_4 - R_3)} \int_{R_1}^{R_2} \int_{R_3}^{R_4} \int_{0}^{\pi} \frac{\cos\theta \, r_I r_{II} \mathrm{d}r_I \mathrm{d}r_{II} \mathrm{d}\theta}{r_Q} \tag{8}$$

where

$$r_Q = \sqrt{r_I^2 + r_{II}^2 - 2r_I r_{II} \cos\theta + z_Q^2}; \quad z_Q = z_2 - z_1.$$

The total numbers of turns of the thin disk coils (pancakes) are N_1 and N_2. I_1 and I_2 are corresponding currents. The mutual inductance (8) can be obtained in a semianalytical form expressed over elliptical integrals of the first and second kind and

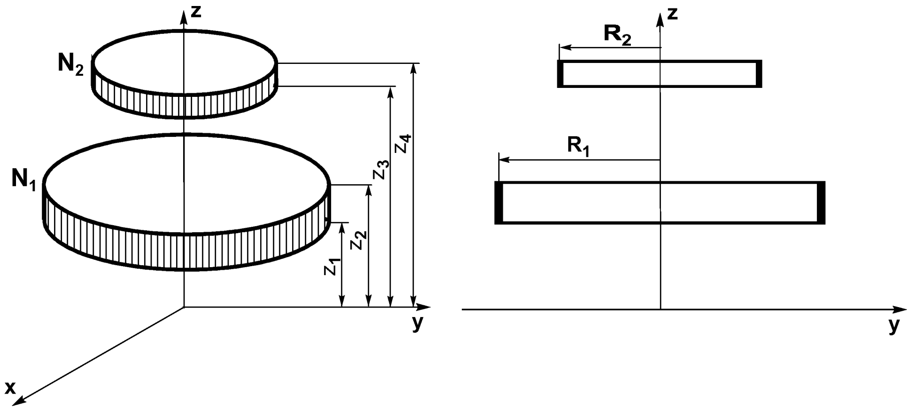

Fig. 2. System: Two thin wall solenoids.

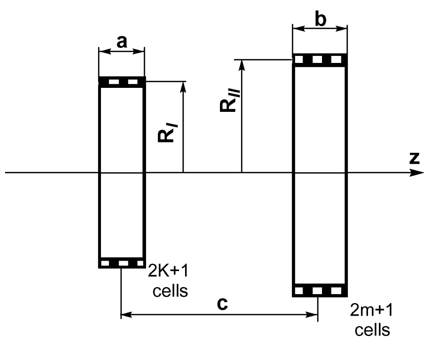

Fig. 3. Configuration of mesh matrix: Two thin wall solenoids.

the Heuman's Lambda function [10]. Applying (1), we obtain the magnetic force for the proposed conductor arrangement in a semianalytical form as

$$F=\frac{\mu_0 N_1 N_2 I_1 I_2}{3(R_2-R_1)(R_4-R_3)}\sum_{n=1}^{4}(-1)^n P_n \tag{9}$$

$$\begin{aligned}
P_n = &-\frac{z_Q k_n^3}{4\sqrt{l_n\rho_n}k_n'^2}\\
&\times\left[z_Q^2-l_n^2-\rho_n^2+\rho_n\sqrt{\rho_n^2+z_Q^2}+l_n\sqrt{l_n^2+z_Q^2}\right]\\
&\times E(k_n)+\frac{k_n\sqrt{l_n\rho_n}}{2}\\
&\times\left[5z_Q+\frac{\rho_n}{z_Q}\sqrt{\rho_n^2+z_Q^2}+\frac{l_n}{z_Q}\sqrt{l_n^2+z_Q^2}-\frac{\rho_n^2}{z_Q}-\frac{l_n^2}{z_Q}\right]\\
&\times K(k_n)-3\pi\frac{\rho_n\sqrt{\rho_n^2+z_Q^2}}{4}sign(z_Q)\\
&\times\left[1-\Lambda_0(\theta_{1n},k_n)-\mathrm{sgn}\left(\sqrt{\rho_n^2+z_Q^2}-l_n\right)\right.\\
&\qquad\left.\times(1-\Lambda_0(\theta_{2n},k_n))\right]-3\pi\frac{l_n\sqrt{l_n^2+z_Q^2}}{4}sign(z_Q)\\
&\times\left[1-\Lambda_0(\theta_{3n},k_n)-\mathrm{sgn}\left(\sqrt{l_n^2+z_Q^2}-\rho_n\right)\right.\\
&\qquad\left.\times(1-\Lambda_0(\theta_{4n},k_n))\right]+\frac{z_Q\rho_n\sqrt{m_n}}{2\Delta_{1n}}\\
&\times\left[E(k_n)-k_n'^2K(k_n)\sin^2(\theta_{1n})\right]\\
&+|z_Q|\sqrt{\rho_n^2+z_Q^2}\frac{k_n^2}{4l_n}\left[E(k_n)-K(k_n)\right]\sin(2\theta_{1n})\\
&-\mathrm{sgn}\left(\sqrt{\rho_n^2+z_Q^2}-l_n\right)\frac{z_Q k_n}{4l_n\Delta_{2n}}\\
&\times\left\{\frac{\sqrt{m_n}}{k_n'^2\left|\sqrt{\rho_n^2+z_Q^2}-l_n\right|}\sqrt{\frac{l_n}{\rho_n}}\right.\\
&\qquad\times\left[E(k_n)-k_n'^2K(k_n)\sin^2(\theta_{2n})\right]\\
&\qquad\times\left[\rho_n(\rho_n-l_n)^2+\rho_n z_Q^2-z_Q^2k_n^2\sqrt{\rho_n^2+z_Q^2}\right]\\
&\qquad+|z_Q|k_n\sqrt{\rho_n^2+z_Q^2}\left[K(k_n)-E(k_n)\right]\\
&\qquad\left.\times\sin(\theta_{2n})\cos(\theta_{2n})\right\}+\frac{z_Q l_n\sqrt{p_n}}{2\Delta_{3n}}\\
&\times\left[E(k_n)-k_n'^2K(k_n)\sin^2(\theta_{3n})\right]\\
&+|z_Q|\sqrt{l_n^2+z_Q^2}\frac{k_n^2}{4\rho_n}\left[E(k_n)-K(k_n)\right]\sin(2\theta_{3n})\\
&-\mathrm{sgn}\left(\sqrt{l_n^2+z_Q^2}-\rho_n\right)\frac{z_Q k_n}{4\rho_n\Delta_{4n}}\\
&\times\left\{\frac{\sqrt{p_n}}{k_n'^2\left|\sqrt{l_n^2+z_Q^2}-\rho_n\right|}\sqrt{\frac{\rho_n}{l_n}}\right.\\
&\qquad\times\left[E(k_n)-k_n'^2K(k_n)\sin^2(\theta_{4n})\right]\\
&\qquad\times\left[l_n(\rho_n-l_n)^2+l_n z_Q^2-z_Q^2k_n^2\sqrt{l_n^2+z_Q^2}\right]\\
&\qquad+|z_Q|k_n\sqrt{l_n^2+z_Q^2}\left[K(k_n)-E(k_n)\right]\\
&\qquad\left.\times\sin(\theta_{4n})\cos(\theta_{4n})\right\}-\frac{3z_Q^2}{2}J_{1n}
\end{aligned}$$

$$J_{1n}=\int_0^{\pi/2} f(\beta)\mathrm{d}\beta$$

$$\begin{aligned}
f(\beta)=\frac{1}{\sin(\beta)}&\left[\tan^{-1}\frac{z_Q^2\cos(\beta)+\rho_n l_n\sin^2(\beta)}{z_Q\sin(\beta)r_{QN}^-}\right.\\
&\left.-\tan^{-1}\frac{z_Q^2\cos(\beta)-\rho_n l_n\sin^2(\beta)}{z_Q\sin(\beta)r_{QN}^+}\right]
\end{aligned}$$

$$r_{QN}^-=\sqrt{l_n^2+\rho_n^2+z_Q^2-2l_n\rho_n\cos(\beta)}$$

$$r_{QN}^+=\sqrt{l_n^2+\rho_n^2+z_Q^2+2l_n\rho_n\cos(\beta)}$$

$$f(0)=\frac{\sqrt{(l_n+\rho_n)^2+z_Q^2}-\sqrt{(l_n-\rho_n)^2+z_Q^2}}{z_Q},$$

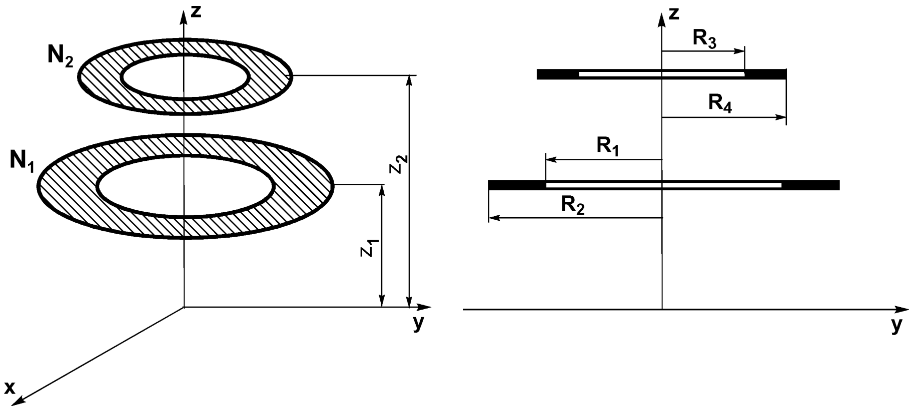

Fig. 4. System: Two thin disk coils (pancakes).

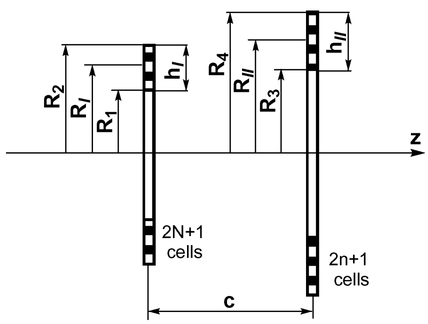

Fig. 5. Configuration of mesh matrix: Two thin disk coils (pancakes).

$$
\begin{aligned}
f(\pi/2) &= 2\tan^{-1}\frac{l_n\rho_n}{z_Q\sqrt{l_n^2+\rho_n^2+z_Q^2}}\\
k_n^2 &= \frac{4\rho_n l_n}{(l_n+\rho_n)^2+z_Q^2},\quad k_n'^2=1-k_n^2\\
m_n &= \frac{2\rho_n}{\sqrt{\rho_n^2+z_Q^2}+\rho_n},\quad p_n=\frac{2l_n}{\sqrt{l_n^2+z_Q^2}+l_n}\\
\theta_{1n} &= \sin^{-1}\frac{|z_Q|}{\sqrt{\rho_n^2+z_Q^2}+\rho_n},\quad \theta_{2n}=\sin^{-1}\sqrt{\frac{1-m_n^2}{1-k_n^2}}\\
\theta_{3n} &= \sin^{-1}\frac{|z_Q|}{\sqrt{l_n^2+z_Q^2}+l_n},\quad \theta_{4n}=\sin^{-1}\sqrt{\frac{1-p_n^2}{1-k_n^2}}\\
\Delta_{1n} &= \sqrt{1-k_n'^2\sin^2\theta_{1n}},\quad \Delta_{2n}=\sqrt{1-k_n'^2\sin^2\theta_{2n}}\\
\Delta_{3n} &= \sqrt{1-k_n'^2\sin^2\theta_{3n}},\quad \Delta_{4n}=\sqrt{1-k_n'^2\sin^2\theta_{4n}}\\
&\rho_1=\rho_2=R_1,\ {}_1=\rho_2=R_2,\ l_1=l_4=R_3,\ l_2=l_3=R_4.
\end{aligned}
$$

If $z_Q = 0$ the axial magnetic force is equal zero. The magnetic force calculation between two thin disks (pancakes) appears for the first time in the literature in the presented form. In order to apply the filament method, the thin disks are then considered to be subdivided into meshes of filamentary coils as shown in Fig. 5. Using the same logic as [14] and [15], we obtain the magnetic force of treated system as

$$
F=\frac{N_1N_2}{(2N+1)(2n+1)}\sum_{p=-N}^{p=N}\sum_{s=-n}^{s=n}F(p,s) \tag{10}
$$

where

$$
\begin{aligned}
F(p,s) &= \frac{\mu_0 I_1 I_2 kz}{4\sqrt{R_I R_{II}}}\left[\frac{(2-k^2)}{(1-k^2)}E(k)-2K(k)\right]\\
R_I(p) &= R_I+\frac{h_I}{2N+1}p;\ p=-N,\ldots,0,\ldots,N\\
R_{II}(s) &= R_{II}+\frac{h_{II}}{2n+1}s;\ s=-n,\ldots,0,\ldots,n\\
h_I &= \frac{R_2-R_1}{2};\ R_I=\frac{R_2+R_1}{2}\\
h_{II} &= \frac{R_4-R_3}{2};\ R_{II}=\frac{R_4+R_3}{2}\\
z=c,\quad k^2(g) &= \frac{4R_I(p)R_{II}(s)}{(R_I(p)+R_{II}(s))^2+z^2}.
\end{aligned}
$$

N_1 and N_2 are the total number of turns of the thin disk coils (pancakes). I_1 and I_2 are currents that correspond to thin disk coils respectively.

C. Thin Wall Solenoid-Thin Disk Coil (Pancake)

The mutual inductance between a thin wall solenoid and a thin disk coil (see Fig. 6) is given by [6] as

$$
M=\frac{\mu_0 N_1 N_2 R}{(R_2-R_1)(z_2-z_1)}\int_{z_1}^{z_2}\int_{R_1}^{R_2}\int_{0}^{\pi}\frac{\cos\theta r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta}{r_Q} \tag{11}
$$

where

$$
r_Q=\sqrt{r^2+R^2-2rR\cos\theta+(z-z_Q)^2}.
$$

The total numbers of turns of the thin wall solenoid and the thin disk coil (pancake) are N_1 and N_2. I_1 and I_2 are corresponding currents. The mutual inductance can be obtained in a semianalytical form expressed over elliptical integrals of the first and second kind and the Heuman's Lambda function, [6]. Applying (1), we obtain the magnetic force for the proposed conductor arrangement in a semianalytical form as

$$
F=\frac{\mu_0 N_1 N_2 I_1 I_2 R^3}{(R_2-R_1)(z_2-z_1)}\sum_{n=1}^{4}(-1)^n P_n \tag{12}
$$

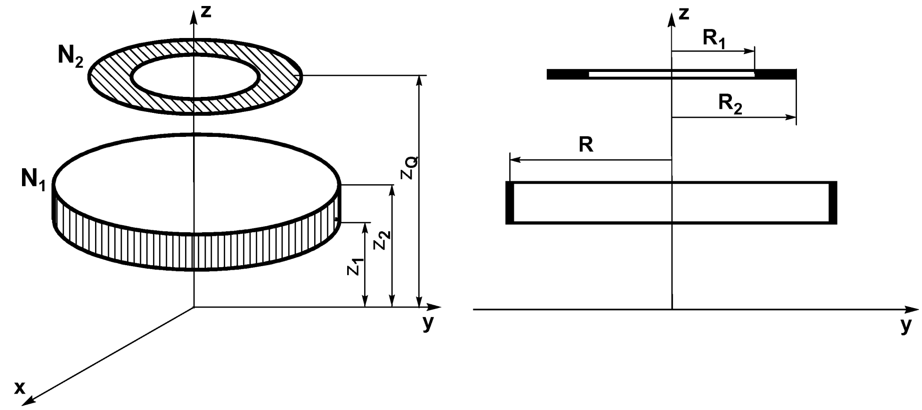

Fig. 6. System: Thin wall solenoid-Thin disk coil (pancake).

where

$$
\begin{aligned}
P_n =& \frac{2\alpha_n\sqrt{\alpha_n}}{3k_n}E(k_n) - \frac{k_n^3\beta_n^2}{24k_n'^2\sqrt{\alpha_n}} \\
&\times\left\{\alpha_n^2+2-3\alpha_n-\frac{\beta_n^2-2}{\sqrt{\beta_n^2+1}+1}-\frac{2\alpha_n^3}{\alpha_n+1}-\beta_n^2\right\} \\
&\times E(k_n)+\frac{k_n\sqrt{\alpha_n}}{6} \\
&\times\left\{\alpha_n^2+4-3\alpha_n-\frac{2\beta_n^2+2}{\sqrt{\beta_n^2+1}+1}\right. \\
&\quad\left.-\frac{2\alpha_n^3}{\alpha_n+1}-2\beta_n^2\right\}K(k_n)+\frac{\pi}{4}|\beta_n|\sqrt{\beta_n^2+1} \\
&\times\left\{1-\Lambda_0(\theta_n,k_n)-\operatorname{sgn}\left(\sqrt{\beta_n^2+1}-\alpha_n\right)\right. \\
&\quad\left.\times\left[1-\Lambda_0(\varepsilon_n,k_n)\right]\right\}-\frac{\alpha_n^2-3}{24\Delta_1}k_n^2 \\
&\times\left\{\left[E(k_n)-k_n'^2K(k_n)\sin^2\varepsilon_n\right]\frac{k_n(1-\alpha_n)}{k_n'^2}\right. \\
&\quad-|\beta_n|\operatorname{sgn}(1-\alpha_n)\left[K(k_n)-E(k_n)\right] \\
&\quad\left.\times\sin\varepsilon_n\cos\varepsilon_n\right\}-\frac{\beta_n^2-2}{12\Delta_2} \\
&\times\left\{\sqrt{m_n}\left[E(k_n)-k_n'^2K(k_n)\sin^2\theta_n\right]\right. \\
&\quad+\frac{|\beta_n|\sqrt{\beta_n^2+1}}{2\alpha_n}k_n^2\left[K(k_n)-E(k_n)\right]\sin\theta_n \\
&\quad\left.\times\cos\theta_n+\frac{k_n\left(\beta_n^2-2\right)\operatorname{sgn}\left(\sqrt{\beta_n^2+1}-\alpha_n\right)}{24\alpha_n\Delta_3}\right. \\
&\times\left\{\frac{\sqrt{\alpha_n}\sqrt{m_n}}{k_n'^2\left|\sqrt{\beta_n^2+1}-\alpha_n\right|}\right. \\
&\quad\times\left[E(k_n)-k_n'^2K(k_n)\sin^2\xi_n\right] \\
&\quad\times\left[(\alpha_n-1)^2+\beta_n^2-k_n^2\beta_n^2\sqrt{\beta_n^2+1}\right]+k_n|\beta_n| \\
&\quad\left.\times\sqrt{\beta_n^2+1}\left[K(k_n)-E(k_n)\right]\sin\xi_n\cos\xi_n\right\} \\
&-J_0(\alpha_n,\beta_n)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_n &= \frac{\rho_n}{R},\ \beta_n=\frac{t_n}{R},\ h_n=\frac{4\alpha_n}{(\alpha_n+1)^2}, \\
k_n^2 &= \frac{4\alpha_n}{(\alpha_n+1)^2+\beta_n^2} \\
m_n &= \frac{2}{\sqrt{\beta_n^2+1}+1},\quad k_n'^2=1-k_n^2, \\
\varepsilon_n &= \sin^{-1}\sqrt{\frac{1-h_n}{1-k_n^2}} \\
\theta_n &= \sin^{-1}\frac{|\beta_n|}{\sqrt{\beta_n^2+1}+1},\quad \xi_n=\sin^{-1}\sqrt{\frac{1-m_n}{1-k_n^2}} \\
\Delta_1 &= \sqrt{1-k_n'^2\sin^2\varepsilon_n}, \\
\Delta_2 &= \sqrt{1-k_n'^2\sin^2\theta_n}, \\
\Delta_3 &= \sqrt{1-k_n'^2\sin^2\xi_n} \\
J_0(\alpha_n,\beta_n) &= \int_0^{\pi/2}\sinh^{-1}\frac{\alpha_n+\beta_n\cos 2x}{\sqrt{1+\beta_n^2\sin^2 2x}}\mathrm{d}x \\
\rho_1 &= \rho_4=R_1,\quad \rho_2=\rho_3=R_2 \\
t_1 &= t_2=z_2-z_Q,\quad t_3=t_4=z_1-z_Q.
\end{aligned}
$$

If $z_Q = 0$ the axial magnetic force is equal zero. The magnetic force calculation between the thin wall solenoid and the thin disk (pancake) appears for the first time in the literature in the presented form.

In order to apply the filament method, the thin wall solenoid and the thin disk are then considered to be subdivided into meshes of filamentary coils as shown in Fig. 7. We obtain a set of thin circular coils (Maxwell's coils). Using the same logic as [14] and [15], we obtain the magnetic force of treated system as

$$
F=\frac{N_1N_2}{(2N+1)(2n+1)}\sum_{g=-K}^{g=K}\sum_{s=-n}^{s=n}F(g,s) \tag{13}
$$

$$
\begin{aligned}
F(g,s) &= \frac{\mu_0 I_1 I_2 kz}{4\sqrt{RR(s)}}\left[\frac{(2-k^2)}{(1-k^2)}E(k)-2K(k)\right] \\
R(s) &= R_s+\frac{h_{II}}{2n+1}s;\ s=-n,\ldots,0,\ldots,n \\
h_{II} &= \frac{R_4-R_3}{2};\ R_I=R;\ R_{II}=\frac{R_4+R_3}{2}
\end{aligned}
$$

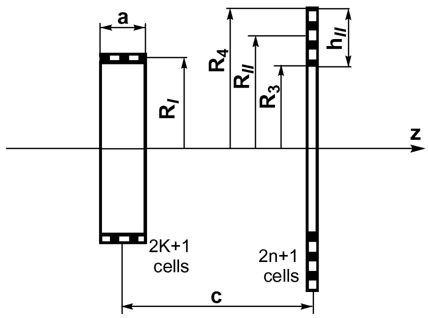

Fig. 7. Configuration of mesh matrix: Thin wall solenoid-Thin disk coil (pancake).

$$z(g) = c - \frac{a}{2K+1}g;\ g = -K, \ldots, 0, \ldots, K$$
$$k^2(g,s) = \frac{4RR(s)}{(R+R(s))^2 + z(g)^2}.$$

N_1 and N_2 are the total number of turns of the thin wall solenoid and the thin disk coil (pancake) respectively. I_1 and I_2 are corresponding currents.

IV. Examples

A. Two Thin Wall Solenoids

Example 1: Calculate the magnetic force between two solenoids with the following dimensions: $R_1 = R_2 = 0.5$ m, $z_1 = 0.3$ m, $z_2 = 0.5$ m, $z_3 = 0.65$ m, $z_4 = 0.95$ m. The corresponding turns and currents are: $N_1 = 100$, $N_2 = 200$, $I_1 = I_2 = 1$ A [3].

The proposed approach (6) yields a magnetic force value of

$$F = 23.83657567265157 \text{ mN}.$$

The execution time was 0.01 s.

By [3], a magnetic force value is

$$F = 23.83657567265133 \text{ mN}.$$

The execution time was 0.12 s. All results are in an excellent agreement.

Example 2: Two solenoids are assumed with the following data:

$R_1 = 0.1$ m, $a = 0.15$ m, $N_1 = 225$; (R_1,a—corresponding radius and height of the first solenoid);

$R_2 = 0.09$ m, $b = 0.15$ m, $N_2 = 300$; (R_2, b-corresponding radius and height of the second solenoid).

The distance between their centers is $s = 0.05$ m. Calculate the force of attraction between solenoids if each solenoid carries one ampere [1].

Proposed method (6):

$$R_1 = 0.1 \text{ m},\ R_2 = 0.09 \text{ m},\ N_1 = 225,\ N_2 = 300,$$
$$I_1 = I_2 = 1 \text{ A},\ z_1 = -0.075 \text{ m},\ z_2 = 0.075 \text{ m},$$
$$z_3 = -0.075 \text{ m},\ z_4 = 0.075 \text{ m}.$$

TABLE I
Comparison of Computational Efficiency

K m	F(mN)	Computational time(s)	Error(%)
All 25	45.81364897472379	0.2303312	0.01981769
All 50	45.80688669361045	0.7711088	0.00505436
All 100	45.80515616109912	2.8841472	0.00127628
All 300	45.80463695568903	26.1075410	0.00014276
All 800	45.80458078097639	178.5066800	0.00002012

Filament method (7):

$$R_I = 0.1 \text{ m},\ R_{II} = 0.09 \text{ m},\ a = b = 0.15 \text{ m},$$
$$N_1 = 225,\ N_2 = 300,\ I_1 = I_2 = 1 \text{ A},\ c = 0.075 \text{ m}.$$

The comparative calculation was made in MATLAB programming.

By [1] a magnetic force value is

$$F = 45.7980 \text{ mN}.$$

The proposed approach (6) yields a magnetic force value of

$$F = 45.80457156637387 \text{ mN}.$$

By [3], a magnetic force value is

$$F = 45.80457156637380 \text{ mN}.$$

Table I shows values of the magnetic force obtained using the filament method. The corresponding computational times and the absolute errors of calculation compared with the exact value from (6) are also given. It can be noted that all results obtained using expressions (6) are in excellent agreement with those obtained using expression (7). Nevertheless, when using the filament method, one has to make more subdivisions, which somewhat increases the computational cost.

Example 3: Assume the case when the solenoids are farther apart [1]:

$$R_1 = 0.25 \text{ m},\ R_2 = 0.2 \text{ m},\ N_1 = 40,\ N_2 = 120,$$
$$I_1 = I_2 = 1 \text{ A},\ z_1 = -0.03 \text{ m},\ z_2 = 0.03 \text{ m},$$
$$z_3 = 0.28 \text{ m},\ z_4 = 0.32 \text{ m}.$$

By [1], a magnetic force value is

$$F = 1.869 \text{ mN}.$$

The proposed approach (6) yields a magnetic force value of

$$F = 1.869154790142725 \text{ mN}.$$

All results are in an excellent agreement.

B. Two Thin Disk Coils (Pancakes)

Example 4: Find the force between two duplicate thin disk coils in air in which a single phase short circuit current of 1.42 effective amperes is following. The coils are on the same axis, [2].

Mean radius: $R = 11.78$ cm.

Radial breadth of the coil winding: $t = 8.32$ cm.

TABLE II
COMPARISON OF COMPUTATIONAL EFFICIENCY

N *n*	*F*(*N*)	*Computational time*(*s*)	*Error*(%)
All 10	2.587794142876992	0.1001440	0.04257631
All 50	2.586740414993551	0.7711088	0.00183982
All 100	2.586704840547111	2.8941616	0.00046454
All 300	2.586694168419037	25.6969504	0.00005196
All 1000	2.586692945640539	314.1517280	0.00000469

Axial spacing: $s = 4.68$ cm.
Number of turns in each coil: $N = 516$.
The measured force [2] was

$$F = 1.16739 \text{ N}.$$

The computed force [2] was

$$F = 1.04967 \text{ N}.$$

By presented approach (9), the computed force was

$$F = 1.050694343958323 \text{ N}.$$

By presented filament approach (10), the computed force was

$$F = 1.05069947929901 \text{ N}.$$

The number of subdivisions was $N = n = 800$.

Example 5:

Proposed method (9):

$$R_1 = 0.16 \text{ m},\ R_2 = 0.28 \text{ m},\ R_3 = 0.11 \text{ m},\ R_4 = 0.26 \text{ m},$$
$$N_1 = N_2 = 100,\ I_1 = 10 \text{ A},\ I_2 = 10 \text{ A},$$
$$z_1 = 0 \text{ m},\ z_2 = 0.05 \text{ m}.$$

Filament method (10):

$$R_I = 0.22 \text{ m},\ R_{II} = 0.185 \text{ m},\ h_I = 0.12 \text{ m},\ h_{II} = 0.15 \text{ m},$$
$$N_1 = N_2 = 100,\ I_1 = 10 \text{ A},\ I_2 = 10 \text{ A},\ c = 0.05 \text{ m}.$$

The proposed approach (9) yields a magnetic force value of

$$F = 2.586692824396309 \text{ N}.$$

The execution time was 0.3405 s.

Table II shows values of the magnetic force obtained using the filament method. The corresponding computational times and the absolute errors of calculation compared with the exact value from (9) are also given. It can be noted again that all results obtained using expressions (9) are in excellent agreement with those obtained using expression (10). The comparative calculation was made in MATLAB programming.

Example 6: In [19], the axial force between two identical coaxial thin disk coils calculated using the calculating functions.$R_1 = R_3 = 12$ cm, $R_2 = R_4 = 23$ cm, $N_1 = N_2 = 100$, $I_1 = I_2 = 10$ A. The distance between disks was $h = 2$ cm.

In [19], two values of the axial force were given:

$$F = 4.150774 \text{ N}$$
$$F = 4.173398 \text{ N}.$$

The first axial force is considered as the accurate value.

Using the presented approach (9), the magnetic force is

$$F = 4.1507739 \text{ N}.$$

TABLE III
COMPARISON OF COMPUTATIONAL EFFICIENCY

K *n*	*F*(*mN*)	*Computational time*(*s*)	*Error*(%)
All 8	0.23118478	0.18999	0.00691174
All 20	0.23119801	1.12099	0.00118944
All 40	0.23120005	4.25600	0.00030709
All 80	0.23120058	16.65400	0.00007785
All 120	0.23120068	37.65400	0.00003460

Thus, the magnetic force obtained by this approach perfectly correspond to the first value given in [19] even though all results are in a very good agreement.

C. Thin Wall Solenoid-Thin Disk Coil (Pancake)

Example 7:

Proposed method (12):

$$R = 0.1 \text{ m},\ R_1 = 0.2 \text{ m},\ R_2 = 0.4 \text{ m},$$
$$z_1 = -0.1 \text{ m},\ z_2 = 0.1 \text{ m},\ z_Q = 0.6 \text{ m},$$
$$N_1 = 100,\ N_2 = 100,\ I_1 = 1 \text{ A},\ I_2 = 1 \text{ A}.$$

Filament method (13):

$$R_I = 0.1 \text{ m},\ R_{II} = 0.3 \text{ m},\ a = 0.2 \text{ m},$$
$$N_1 = 100,\ N_2 = 100,\ I_1 = 1 \text{ A},\ I_2 = 1 \text{ A},\ c = 0.6 \text{ m}.$$

The proposed approach (12) yields a magnetic force value of

$$F = 0.23120076 \text{ mN}.$$

Execution time was 0.04999 s.

Table III shows values of the magnetic force obtained using the filament method. The corresponding computational times and the absolute errors of calculation compared with the exact value from (12) are also given. It can be noted again that all results obtained using expressions (12) are in excellent agreement with those obtained using expression (13).

In all examples, we used the filament method as a comparative method. Obviously, the calculation results vary depending on the number of meshes in the coils (the number of filaments). However, the changes of the calculation results become smaller and smaller as the number of meshes (filaments) is increased. In order to save calculation time, one has to limit the number of meshes (filaments) so that a desired accuracy would be needed. From all previously solved problems, it is clear that one does not need to take a lot of filaments to obtain a desired accuracy. From the engineering point of view, only four decimal digits are more then enough so that we do not need to take enormous filaments to obtain a satisfactory accuracy. This statement has been proved by previously solved examples.

V. CONCLUSION

This paper presents new analytical and semianalytical expressions for the calculation of magnetic forces between thin circular coils (two wall solenoids, two thin disk coils and thin wall solenoid-thin disk coil). Results obtained by these new expressions have been compared with the well-known filament method and already published data. The results have demonstrated better accuracy and lower computation cost. All programs were written in MATLAB programming, showing that the magnetic force can be efficiently calculated with only a personal computer. According to our knowledge, the calculation of the magnetic force between two disks (pancakes) and between the thin wall solenoid and the thin disk coil appears for the first time in the literature. The presented method is suitable either for microcoils or large coils.

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