University of Central Florida
Online Mathematics Contest
Problems: April 2014 (Year 1, Round 3)

Warm Ups

1) Alan, Betty, Carol and David are running a gift-exchange amongst themselves. In how many ways can they give their gifts assuming that each of them must give a gift to a different person in the group and each person in the group must receive exactly one gift?

2) How many divisors does the number 46800 have?

3) Find the sum of all positive rational numbers that are less than 5 and that have a denominator of 36, when written in lowest terms.

4) Prove that the equation $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$ has no solutions in positive integers x, y and z.

5) Find all positive integers n such that $n^2 + 1$ is divisible by $n + 1$.

Exercises

1) Prove for all integers $n \geq 4$ that $n! > n^2$.

2) Alex needs to create a new password for his email account. His password must be exactly 8 characters long and each character must come from a set of 96 printable characters. Furthermore, his password must satisfy the following restrictions:

a) It must contain at least one upper case letter, one lower case letter, one digit and one special character. Note: Every character that is NOT a letter or digit is considered a special character.

b) No substring of his password can contain his name, regardless of case. So, “AlEX”, “ALEX”, “alex” and “AleX” are all examples of substrings that are not allowed in any valid password.

How many possible passwords can Alex create?

3) Prove that every integer $n > 6$ can be expressed as the sum of two relatively prime integers that are both greater than 1.

4) Solve the following system of equations for x, y and z in terms of the constants a, b and c:

$$x + y = a \qquad x + z = b \qquad y + z = c$$

5) A palindromic number is one with a leading digit greater than 0 that is the same read forwards and backwards. How many different palindromic numbers of exactly 11 digits are divisible by 3?

Investigations

1) A baseball team plays 45 games in 30 days. The team plays at least one game every day. Prove that there is some consecutive window of days in which the team plays exactly 14 games.

2) Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with all real roots. Prove that $2a^2 - 3b \geq 0$.

3) Imagine flipping a biased coin, with a probability of landing heads equal to $\frac{3}{4}$, 2n times. What is the probability that it will land heads an even number of times, in terms of n?

4) Let a, b and c be positive integers. Let gcd(a, b, c) represent the greatest common divisor of a, b and c and lcm(a, b, c) be the least common multiple of a, b and c. Prove that

$$\frac{(\gcd(a,b,c))^2}{\gcd(a,b)\gcd(a,c)\gcd\,(b,c)} = \frac{(\mathrm{lcm}(a,b,c))^2}{lcm(a,b)lcm(a,c)lcm(b,c)}$$

Note: This problem was taken from the 1972 USA Mathematics Olympiad.

I have chosen it because I think it's a classical problem illustrating the Fundamental Theorem of Arithmetic.

5) Let the sets $A = \{z: z^p = 1\}$ and $B = \{w: w^q = 1\}$, where gcd(p, q) = 1. Both A and B are sets of complex roots of unity. Prove that the set $C = \{zw : z \in A \text{ and } w \in B\}$ is a set of complex roots of unity of size pq.

Directions for submission: If you have not done so yet, please register:

http://eecs.ucf.edu/ucf-omc/register.php

Please either email your solutions in a .pdf attachment to ucfomc@gmail.com with the subject "UCF-OMC April 2014 Submission – Name", where you place your first and last name in place of Name. Alternatively, mail a hard-copy of your solutions (if you do this, please retain the originals just in case something gets lost in the mail) to

UCF-OMC
Attn: Arup Guha
4328 Scorpius Street
Orlando, FL 32816

The deadline for receipt for either method is Friday, April 25, 2014. Good luck!

UCF-OMC Staff