University of Central Florida
Online Mathematics Contest
Problems: February 2014 (Year 1, Round 2)

Warm Ups

1) Let x be the length of the diagonal of a rectangle. If the ratio of the length to the width of the rectangle is 4, what is the perimeter of the rectangle in terms of x?

2) Determine the solutions for t in the following inequality: $26 \leq 90 - e^t \leq 54$.

3) Find a degree 4 (quartic) polynomial function f(x) such that f(1) = 0, f(3) = 0 and f(x) ≥ 0, for all real x.

4) The current in a river is flowing at 3 mph downstream. Without any current, Serena rows her kayak at 5 mph. She is planning a trip where she'll go downstream and then return to where she started, going back upstream. How far should she go downstream before turning around so that she returns to her starting spot exactly 2 hours after she left it?

5) Josiah is making hot chocolate for Jasmine from premade mixtures of milk and chocolate. Jasmine prefers her hot chocolate to have 6 parts milk for 1 part chocolate. Unfortunately, Josiah only has access to two mixtures of hot chocolate, neither of which is mixed to the ratio that Jasmine prefers. Mixture A has 3 parts milk for 1 part chocolate and Mixture B has 10 parts milk for 1 part chocolate. If Josiah wants to make Jasmine 8 ounces of hot chocolate according to her preference, how many ounces of mixture A should he use and how many ounces of mixture B should he use? (Note: The sum of your answers must be 8.)

Exercises

1) Define a simple integer to be one that is divisible by 2, 3 or 5. How many simple integers are less than or equal to 1000000?

2) What is the maximum integer k for which 1000! is divisible by 10^k?

3) How many non-negative integer solutions are there to the equation a + b + c + d = 20, such that a < 10 and b > 5?

4) For all positive integers n, prove $2\left(\sqrt{n+1} - \sqrt{n}\right) < \frac{1}{\sqrt{n}}$.

5) Let r and s be the roots of the quadratic function $f(x) = x^2 - 9x + 16$. Determine the quadratic function with leading coefficient 1 with roots r^3 and s^3.

Investigations

1) Prove, using basic geometric facts and Heron's formula for triangle area, prove that the inradius of a triangle with sides a, b and c, with $s = \frac{a+b+c}{2}$ is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

2) Determine the following infinite sum: $\sum_{i=0}^{\infty} i^2 p^i$, where $0 < p < 1$, in terms of p.

3) Let T be a isosceles triangle with a base length of a, and two sides of length b. Let C_1 be the incircle of T. Let C_i be the circle inscribed between the two sides of length b and C_{i-1}, for all $i > 1$. Determine $\sum_{i=1}^{\infty} Area(C_i)$.

4) A dice game is played as follows: Roll a pair of fair six-sided dice. If the sum of the dice rolled is 12, you win. If not, roll the pair again. If this total is less than or equal to the previous total rolled, you lose. Continue rolling until either you lose because your next roll is less than or equal to the previous roll, or your total is 12, in which case you win. Given that Samantha won the game, what is the expected number of times she rolled the dice?

5) Let n be a positive integer. Determine $\sum_{i=1}^{n} \left[\left(\cos\left(\frac{2\pi i}{2n+1}\right) - 1\right)\left(2\cos\left(\frac{2\pi i}{2n+1}\right) + 1\right)\right]$.

Directions for submission: If you have not done so yet, please register:

http://eecs.ucf.edu/ucf-omc/register.php

Please either email your solutions in a .pdf attachment to ucfomc@gmail.com with the subject "UCF-OMC February 2014 Submission – Name", where you place your first and last name in place of Name. Alternatively, mail a hard-copy of your solutions (if you do this, please retain the originals just in case something gets lost in the mail) to

UCF-OMC
Attn: Arup Guha
4328 Scorpius Street
Orlando, FL 32816

The deadline for receipt for either method is Friday, March 7, 2014. Good luck!

UCF-OMC Staff