University of Central Florida
Online Mathematics Contest
Problems: February 2015 (Year 2, Round 3)

Warm Ups

1) The ratio of the shorter side of a rectangle to its longer side is equal to the ratio of the longer side of the rectangle to the sum of the shorter and longer side. If the area of the rectangle is $2 + 2\sqrt{5}$, what are the lengths of the sides of the rectangle?

2) We select a real number, *x*, at random from the range (0, 1). Then we select a real number, *y*, at random from the range (*x*, 1). What is the expected value of *y*?

3) An item is repeatedly discounted by 10%. How many times does it have to be discounted before the effective discount from the original price is at least 90%? (Note: derive an expression to your answer without a calculator. You may you the calculation as your final step, to find the numerical value of your expression.)

4) Shemina flips a fair coin n times in a row. What is the probability that she never sees the same outcome on two consecutive tosses?

5) The sum of five terms of an arithmetic sequence is 50 and the product of those five terms is 58240. What are the five terms of the sequence?

Exercises

1) Students are sitting for an exam in desks arranged in one long row. There are a total of 17 desks and 5 students taking the exam. The professor would like the students arranged such that there is at least one empty desk in between any pair of students. How many different sets of desks can the students sit in that satisfy this constraint?

2) Prove that the system of equations below has no solutions for positive real values for a, b and c:

$$a^2 + b^2 = c^2$$
$$\sqrt{a} + \sqrt{b} = \sqrt{c}$$

3) Find all positive integer solutions (x, y, z, w) with x ≤ y ≤ z ≤ w such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} = 1$.

4) A three digit integer x is added to a four digit integer y and no carrying is required. How many possible values are there for the ordered pairs (x, y)?

5) Find the area of the largest regular hexagon that fits inside a square of side length 1.

Investigations

1) a. Show that 2015 is a difference of two cubes.

b. What is the next year that is a sum or difference of two cubes?

c. What was the most recent previous year that was a sum or difference of two cubes?

2) Suppose that $a_1, a_2, \ldots, a_n$ is a permutation of 1, 2, ... , n .

a. What's the smallest possible value of $|a_1 - a_2| + |a_2 - a_3| + \cdots + |a_{n-1} - a_n| + |a_n - a_1|$?

b. How many permutations achieve this minimum value?

3) A bag contains n pairs of distinct matching tiles. We then start pulling out tiles from the bag at random, one by one. Any time we pull the matching tile to one of the tile in our hand, we set aside the match. The game stops if we are ever holding three non-matching tiles at the end of a turn, or if we've pulled all of the tiles from the bag. If the latter occurs, we've won the game. What's the probability of winning the game, in terms of n?

4) Find the set of integers n that satisfy the following equation: $\sum_{i=45}^{133} \frac{1}{\sin(i^\circ)\sin((i+1)^\circ)} = \frac{1}{\sin(n^\circ)}$

5) Let the set N_d be the set of divisors of the positive integer N. With proof, determine $\sum_{k \in N_d} \phi(k)$, where ϕ represents the Euler phi function.

Please either email your solutions in a .pdf attachment to ucfomc@gmail.com with the subject "UCF-OMC February 2015 Submission – Name", where you place your first and last name in place of Name.

The deadline for receipt for either method is Friday, March 6, 2015. Good luck!

UCF-OMC Staff