**University of Central Florida**
**Online Mathematics Contest**
**Problems: March 2015 (Year 2, Round 4)**

Warm Ups

1) If $log_x y = 2log_z y$ what is $log_x z$? Assume that x, y and z are all greater than 1.

2) Given that $z = -2 + 7i$ is a root to the equation: $z^3 + 6z^2 + 61z + 106 = 0$, what are the other two roots of the equation?

3) Find all the points of intersections of the circle $x^2 + 2x + y^2 + 4y = -1$ and the line $x - y = 1$.

4) Rubba and Prince decided that they were going to have a contest who could get the most money in March. Each one of them came up with a different way to beat the other. At the end of the month, Rubba ended up doing several odd jobs around Orlando and made $10 million dollars. Prince, on the other hand made a deal with the university that he should get a penny one day, and double the amount of pennies the next day, and then double the pennies of previous day, continually for each day in March. Who made the most money in March?

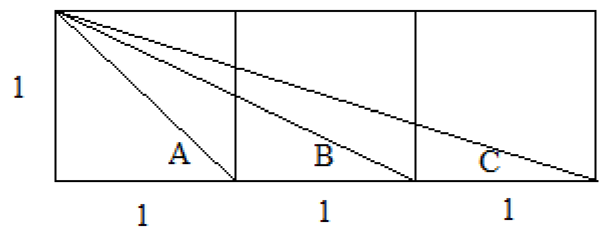
5) Alaina's age uses the same two digits as Brett's age, but the two digits are in reverse order in her age. The difference between the square of Alaina and Brett's ages is also a perfect square. If Alaina is older than Brett, how old are they both?

Exercises

1) William is making origami and currently has a paper equilateral triangle, XYZ with side length 8. He folds the triangle so that vertex X touches a point on the side YZ, with a distance 6 from point Y. What is the length of the fold?

2) A number with 3 digits in consecutive ascending order is multiplied by another 3-digit number with digits in consecutive descending order. The product of these two numbers is 110745. What are the two numbers?

3) Consider the diagram with three unit squares side by side below:




where A, B and C are the acute angles denoted. What is the sum of angles A, B and C?

4) Without using a calculator, determine with proof, the exact value of

$$(1 + tan20°)(1 + tan25°).$$

5) Prove that $\sum_{i=0}^{n}\binom{n}{i} = \binom{2n}{n}$, for all positive integers n. (Note: You may find a combinatorial argument more elegant than an algebraic proof.)

Investigations

1) Let S be a subset of {1, 2, 3, 4, ..., 50} such that the difference between each pair of items in S is distinct. With proof, what is the largest possible size of S? Give a set of this size that satisfies the requirement. (Note: the set {1, 3, 7, 15} satisfies the difference requirement for S since the 3 - 1 = 2, 7 - 1 = 6, 15 - 1 = 14, 7 - 3 = 4, 15 - 3 = 12 and 15 - 7 = 8, and all of these differences, 2, 6, 14, 4, 12 and 8 are all different.)

2) Orlando, Florida (USA) is located at 28.4158º N latitude and 81.2989º W longitude (according to Google) while Melbourne, Australia is located at 37.8136º S latitude and 144.9631º E longitude (according to the same source). Assuming that the Earth is a sphere with radius 6371 kilometers, what's the length of the shortest possible plane flight between the two cities, assuming that the plane stays 10 km above the ground for the whole trip. For simplicity, just count the distance once the plane has elevated 10 km above the ground. Assume that the plane rises to this elevation at the two latitude/longitude pairs given in the problem.

3) A bracket for a single elimination tournament between eight seeded teams looks like this:

| | | | | |
|---|---|---|---|---|
| ___(1)___ | | | | ___(2)___ |
| vs | ___(W 1v8)___ | | ___(W 2v7)___ | vs |
| ___(8)___ | | | | ___(7)___ |
| | vs. | _(W left)_ vs _(W right)_ | | |
| ___(4)___ | | | | ___(3)___ |
| vs | ___(W 4v5)___ | | ___(W 3v6)___ | vs |
| ___(5)___ | | | | ___(6)___ |

Seeds represent pre-rankings for teams with the first seed being the best team and the eighth seed being the worst team. The four first round games are denoted on the left and right ends of the bracket. The winners of the two games on each side of the bracket then play each other. Finally the winners of those two games play each other for the championship. For a given tournament assume that when the $i^{th}$ seeded team plays the $j^{th}$ seeded team, the chance of the $i^{th}$ seeded team winning is $\frac{j}{i+j}$. What is the probability that the first seeded team wins the whole tournament?

4) With proof, determine the value of the sum $\sum_{i=1}^{n}(-1)^{i+1}i^2$, in terms of n.

5) Prove that $\prod_{i=1}^{n}\cos\left(\frac{\alpha}{2^i}\right) = \frac{sin\alpha}{2^n \sin(\frac{\alpha}{2^n})}$, for all positive integers n and positive acute angles α.

**Please either email your solutions in a .pdf attachment to ucfomc@gmail.com with the subject "UCF-OMC March 2015 Submission – Name", where you place your first and last name in place of Name.**

**The deadline for receipt for either method is <u>Friday, April 10, 2015.</u> Good luck!**

**UCF-OMC Staff**