University of Central Florida
Online Mathematics Contest
Problems: November 2014 (Year 2, Round 1)

Warm Ups

1) John takes 75 minutes to mow one acre and Sally takes 60 minutes to mow one acre. How many minutes would it take them working together to mow a 6 acre lot?

2) A class has n students in it. Their average test grade was 78. When Alice's grade is removed from the group, the remaining students had an average test grade of 73. What is the maximum value of n for which this information is plausible? For this value of n, what must Alice's test score be?

3) The 15^{th} term in an arithmetic sequence is 68. If the sum of the first twenty terms of the sequence is 1000, what is the value of the first term of the sequence?

4) Consider writing the positive integers in increasing order. What would be the 2014^{th} digit written? (For example, the 20^{th} digit written would be 1, since there are 9 digits in 1 – 9 and 10 digits in 10 – 14, so the 20^{th} digit would be the 1 in writing 15.)

5) What is the value of $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$?

Exercises

1) How many positive integers are divisors of exactly two of the three numbers 15^7, 18^5 and 20^6?

2) How many zeroes are at the end of 2014!?

3) Determine the following sum in terms of n: $\sum_{i=1}^{2^n}\lfloor log_2 i\rfloor$. Note: $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

4) If the graphs of $y = 2|x - a| + b$ and $y = -2|x - c| + d$ intersect at both (5, 6) and (7, 8), what is the value of $a - b + c - d$?

5) During a particular tennis tournament, Sarah won three matches and lost none. These three matches increased her winning percentage by precisely 2. Determine the number of matches Sarah had won prior to the tournament, assuming that she had previously won at least one match. Is it possible to determine the total number of matches she had played prior to the tournament?

Investigations

1) In the xy-plane, what is the length of the shortest path from (0, 0) to (12, 13) that does not go inside the circle $(x - 6)^2 + (y - 8)^2 = 25$?

2) Let m and n be positive integers with m > n. Prove that $m^2 - n^2$, 2mn and $m^2 + n^2$ form a Pythagorean Triple. Determine the least set of further constraints on m and n that guarantees that the Pythagorean Triple designated is a primitive Pythagorean Triple. Note: a primitive Pythagorean Triple is one where the greatest common divisor of the three side lengths is 1.

3) Simplify the expression

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\cdots+\sqrt{1+\frac{1}{2013^2}+\frac{1}{2014^2}}$$

4) What is the smallest non-negative integer of the form

$$\pm 1^3 \pm 2^3 \pm 3^3 \pm \cdots \pm 2014^3,$$

for some choice of signs? Provide proof of this minimum as well as one choice of signs that satisfies it.

5) (a) Suppose that a power of 2 contains the substring 2014. What is the fewest possible number of digits after the '2014'?

(b) Is it possible for a power of 2 to begin with the four digits 2014?

Directions for submission: If you have not done so yet, please register: http://eecs.ucf.edu/ucf-omc/register.php

Please either email your solutions in a .pdf attachment to ucfomc@gmail.com with the subject "UCF-OMC November 2014 Submission – Name", where you place your first and last name in place of Name. Alternatively, mail a hard-copy of your solutions (if you do this, please retain the originals just in case something gets lost in the mail) to

UCF-OMC
Attn: Arup Guha
4328 Scorpius Street
Orlando, FL 32816

The deadline for receipt for either method is Friday, November 21, 2014. Good luck!

UCF-OMC Staff