University of Central Florida
Online Mathematics Contest
Solutions: April 2014 (Year 1, Round 3)

Warm Ups

1) Alan, Betty, Carol and David are running a gift-exchange amongst themselves. In how many ways can they give their gifts assuming that each of them must give a gift to a different person in the group and each person in the group must receive exactly one gift?

Solution
We can solve this problem by brute force, simply listing the number of permutations of ABCD where no letter is in its original place:

BADC, BCDA, BDAC, CADB, CDAB, CDBA, DABC, DCAB and DCBA.

Each of the ways listed corresponds to one way of giving presents. For example, CADB represents Alan buying Carol's present, Betty buying Alan's present Carol buying David's present and David buying Betty's present.

Thus, there are 9 such ways in which the group can do their gift exchange.

More generally, if we have n people doing the gift exchange, we can solve our problem as follows:

Let D_n represent the number of ways in which n people can exchange gifts so that no one buys their own gift. Consider adding an n+1 person to the gift exchange. Since he can't give his gift to himself, one of two things must happen:

1) He receives a gift from the person who gets his gift.
2) He receives his gift from a different person than the one to whom his gift went to.

If we are in case #1, essentially he has exchanged his gift with one person, and there are n-1 people left who need to do the gift exchange. Since he can choose any one of the n original people to swap presents with, the total number of ways in which case #1 occurs is nD_{n-1}.

If we are in case #2, pretend that the new person exchanges his present with one of the original people in the group. But, now, since we've already counted this case, the original person who received the present from the new person can't keep the present, he must give it up! But, this is just like the original gift exchange with n people. Thus, this occurs in nD_n ways.

Summing, we get the following recurrence relation:

$D_{n+1} = nD_n + nD_{n-1}$

It turns out that this problem is a fairly famous combinatorics problem known as derangements. The recurrence above can be used to solve for the n^{th} derangement iteratively. In this specific problem, let's look at the breakdown of the recurrence into the two cases:

Treat ABC as the original people and D as the new person. Let D and A trade presents:

D _ _ A

From here, we must derange B and C, which can only be done in one way:

DCBA.

Switching presents between D and B and D and C yields the two following derangements:

CDAB and BADC.

Now, consider case 2. Let A and D switch presents, but this time, force A to also switch his new present D with either B or C:

BCDA, CDBA.

The two derangements created by having D swap with B and then forcing B to give up her present are:

DCAB, ACDB

Finally, the two derangements created by having D swap with C and then forcing C to give up her present are:

DABC, BDAC.

Read more about derangements at these sites:

http://en.wikipedia.org/wiki/Derangement

http://www.jamestanton.com/wp-content/uploads/2010/12/derangements-essay2.pdf

2) How many divisors does the number 46800 have?

Solution

Let d(n) be the number of divisors for the natural number, n, where $n = a^p b^q c^r \ldots$

All divisors of n must take the form $a^{p1} b^{q1} c^{r1} \ldots$, where $0 \le p1 \le p$, $0 \le q1 \le q$, $0 \le r1 \le r$, etc.

Each unique divisor can be expressed as a solution for each of the given exponents: p1, q1, r1, etc. Thus, we must simply count the number of combinations of settings for each of these exponents. Consider the exponent p1. It must equal 0, 1, 2, ..., or p. Thus, there are p+1 choices for the exponent for the base a. Since the choice of the other exponents is independent of this choice, to get the total number of possible choices, we must multiply each of these expressions to arrive at a formula for d(n):

$d(n) = (p+1)(q+1)(r+1)\ldots$

where a, b and c are the prime factors of n.

Since $46800 = 2^4 3^2 5^2 13^1$, it follows that the number of divisors of 46800 is

$(4 + 1)(2 + 1)(2 + 1)(1 + 1) = 90.$

3) Find the sum of all positive rational numbers that are less than 5 and that have a denominator of 36, when written in lowest terms.

Solution

In order to solve this problem, we need not include the even numbers between 1 and 36 since it will reduce. Let's find the rational numbers between 0 and 1:

1/36, 5/36, 7/36, 11/36, 13/36, 17/36, 19/36, 23/36, 25/36, 29/36, 31/36, 35/36

This sequence will continue until we reach 179/36 since we are not including 5. The sum of these 12 terms is 6 and if we add 1 to each respective term we will acquire the amount of terms from 1 to 2 which would give us another 12 terms. So, if we multiply the sum by the number of sets: 0 to 1, 1 to 2, 2 to 3, 3 to 4, 4 to 5, we acquire the sum of the repeated first set. Using this fact, we will add the product of the number of terms by the sum of each successive set as follows:

$$6(5) + 12(1 + (1+1) + (1+1+1) + (1+1+1+1)) = 30 + 12(10) = 150$$

Therefore, the sum of all the positive rational numbers less than 5 and that have a denominator of 36 is 150.

Consider generalizing the problem for an arbitrary denominator n with a maximum limit of k. Notice that in our specific example, for all of the fractions from 0 to 1, the fractions can be paired up where each pair equals 1. This isn't a coincidence. If gcd(a, n) = 1, gcd(n - a, n) = 1 also, via Euclid's Algorithm. Thus, if $\frac{a}{n}$ is a fraction in lowest terms, $\frac{n-a}{n}$ is also a fraction in lowest term, and vice versa. Thus, if we want to sum up these fractions in between 0 and 1, it's enough for us to know the number of values in between 1 and n that don't share a common factor with n. This value comes up so often Euler coined a function for it: φ(n). For example, φ(36) = 12, since 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31 and 35 are relatively prime with 36. Thus, our desired sum for the fractions in between 0 and 1, in general is $\frac{\phi(n)}{2}$. There are k sets of these fractions in the intervals (0, 1), (1, 2), ..., (k-1, k). Adding the fractional parts of each of these fractions, we obtain the sum $\frac{k\phi(n)}{2}$. Finally, we must take into account the integer parts of the fractions in the intervals (1, 2), (2, 3), ..., (k - 1, k). In general, the integer part in the interval (i, i+1) would sum to iφ(n). Thus, we must sum over all integers i described in the intervals above: $\sum_{i=1}^{k-1} i\phi(n) = \frac{(k-1)k\phi(n)}{2}$.

The total sum is $\frac{k\phi(n)}{2} + \frac{(k-1)k\phi(n)}{2} = \frac{k\phi(n)}{2}(1 + k - 1) = \frac{k^2\phi(n)}{2}$.

Note: Given the prime factorization of n, there's a formula to calculate φ(n), but the proof of this formula is fairly involved.

Here is some reading on the Euler Phi function and Euclid's Algorithm:

http://en.wikipedia.org/wiki/Euler's_totient_function

http://www.math.wisc.edu/~josizemore/Notes11(phi).pdf

http://en.wikipedia.org/wiki/Euclidean_algorithm

http://www.cs.ucf.edu/~dmarino/ucf/cot3100h/lectures/COT3100Euclid01.doc

4) Prove that the equation $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$ has no solutions in positive integers x, y and z.

Solution

Without loss of generality, let x ≥ y. Then, $\frac{x}{y} \geq 1$. Clearly, if x, y and z are positive integers, it follows that both fractions $\frac{y}{z}$ and $\frac{z}{x}$ are positive, thus $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 1 + \frac{y}{z} + \frac{z}{x} > 1$.

Note: If x = y = z, then the sum of the fractions given is 3. This is the minimum value of the sum of the fractions for all positive real x, y and z.

5) Find all positive integers n such that $n^2 + 1$ is divisible by n + 1.

Solution

Use long division to obtain: $\frac{n^2+1}{n+1} = n - 1 + \frac{2}{n+1}$. Since n is an integer, in order for this fraction to be an integer, (n+1) | 2. This only occurs for n = 1. Thus, the only positive integer n for which $n^2 + 1$ is divisible by n + 1 is n = 1.

Exercises

1) Prove for all integers $n \geq 4$ that $n! > n^2$.

Solution

Use induction on n.

Base case: $n = 4$. LHS = $4! = 24$, RHS = $4^2 = 16$, Since $24 > 16$, the base case holds.

Inductive hypothesis: Assume for an arbitrary integer $k \geq 4$ that $k! > k^2$.

Inductive step: Prove for $n = k+1$ that $(k+1)! > (k+1)^2$.

$(k+1)! = (k+1)k!$
$> (k+1)k^2$
$> (k+1)(2k)$, since $k \geq 4$, it follows that $k > 2$.
$> (k+1)^2$, since $2k > k+1$ for all $k \geq 4$.

This proves the assertion for all positive integers $n \geq 4$, as desired.

2) Alex needs to create a new password for his email account. His password must be exactly 8 characters long and each character must come from a set of 96 printable characters. Furthermore, his password must satisfy the following restrictions:

a) It must contain at least one upper case letter, one lower case letter, one digit and one special character. Note: Every character that is NOT a letter or digit is considered a special character.

b) No substring of his password can contain his name, regardless of case. So, “AlEX”, “ALEX”, “alex” and “AleX” are all examples of substrings that are not allowed in any valid password.

How many possible passwords can Alex create?

Solution

This might be one of the ugliest counting problems I've encountered. I should have tried to solve it before I put it in the problem set. Here is the general method of attack:

1) Use the Inclusion-Exclusion Principle to count all strings of 8 letters that contain at least one item from each of the categories listed.

2) Subtract out of the number of strings found in part 1, all strings with a substring of any of the 16 versions of alex that ALSO have all four types of characters.

The first part is quite tedious but isn't awful conceptually, but the second part has a number of unique special cases to consider.

We have four sets of character: 26 lowercase letters, 26 uppercase letters, 10 digits, and 34 special characters. Call these sets A, B, C and D, respectively. Let the set A^k be the set of strings of length k where each letter is taken from set A. Let X be the set we desire. Using the inclusion exclusion principle, we have the following:

$$|X| = |(A \cup B \cup C \cup D)^8| - |A^8| - |B^8| - |C^8| - |D^8|$$
$$+|(A \cup B)^8| + |(A \cup C)^8| + |(A \cup D)^8| + |(B \cup C)^8| + |(B \cup D)^8| + |(C \cup D)^8|$$
$$-|(A \cup B \cup C)^8| - |(A \cup B \cup D)^8| - |(A \cup C \cup D)^8| - |(B \cup C \cup D)^8|$$

Applying the idea that the number of strings of length n where C letters are possible for each slot, creates C^n possible strings, we find that the number of strings that satisfy the first requirement are:

$$96^8 - 26^8 - 26^8 - 10^8 - 34^8 + 52^8 + 36^8 + 60^8 + 36^8 + 60^8 + 44^8 - 62^8 - 86^8 - 2(70^8)$$

Note: I took a shortcut on the last term to make it fit in a single line. The last two terms are both 70^8.

Now, of all of these strings of 8 characters we counted, we have to subtract out the strings that have any version of alex as a subtring. The key category to subdivide our counting is that some versions of alex have both lower and upper case letters (14 to be precise) and some (2) only have one type of letter.

There are 14 versions of alex (take all 16 versions and subtract out both "alex" and "ALEX") that both have lower and uppercase characters. Let's deal with these first. The string alex can be located in any one of five positions (assume a mixed case, not what's written):

A L E X _ _ _ _ or _ A L E X _ _ _ or _ _ A L E X _ _ or _ _ _ A L E X _ or _ _ _ _ A L E X

For each of these 5 positions, it can take on 14 specific versions that alter case so that at least one lower and uppercase letter are required. In the remaining four characters, we must have at least one digit and one special character. This boils down to finding the number of strings of length 4 that contain at least one digit and one special character. The total number of strings of length 4 is 96^4. From these, we must subtract out all the strings that don't have one digit or one special character. There are $86^4 + 62^4 - 52^4$ strings that have either one digit, or a special character, or both. (We applied the inclusion-exclusion principle to the two sets in question to obtain this.) Thus, the number of four character strings that have at least one digit and one special character is $96^4 - (86^4 + 62^4 - 52^4) = 96^4 - 86^4 - 62^4 + 52^4$. It follows that the total number of strings with alex in any of the five positions with mixed case is $5 \times 14 \times (96^4 - 86^4 - 62^4 + 52^4)$, where 5 is the number of different locations for alex, 14 is the number of different capitalizations for alex and the last term is how many ways to fill the open four slots.

Now, we must redo this count for the two cases, ALEX and alex, that only contain one type of letter. Once again, we have a total of 96^4 ways to fill the open slots. But, this time, we must include at least one digit, one special character and one letter of the type not seen in the string alex. Thus, we will subtract out all strings that are missing any of these types of characters, once again, using the Inclusion-Exclusion Principle. We get the following for the number of strings of length four that are missing a character of one of these types.

$86^4 + 62^4 + 70^4 - 52^4 - 60^4 - 36^4 + 26^4$

Thus, the number of strings where alex appears in the same capitalization that HAS all four character types is

$5 \times 2 \times (96^4 - (86^4 + 62^4 + 70^4 - 52^4 - 60^4 - 36^4 + 26^4)) =$

$10 \times (96^4 - 86^4 - 62^4 - 70^4 + 52^4 + 60^4 + 36^4 - 26^4)$

Finally, one might think that we have a problem with double counting strings that contain "alex" twice, but we don't! The reason is that any string that has alex in it twice doesn't have any special characters, and these strings weren't counted at all, to begin with!

Thus, the number of strings we have to subtract out from our original count that have alex as a subtring are:

$5 \times 14 \times (96^4 - 86^4 - 62^4 + 52^4) + 5 \times 2 \times (96^4 - 86^4 - 62^4 - 70^4 + 52^4 + 60^4 + 36^4 - 26^4)$

Our final answer, for number of permissible passwords is:

$$96^8 - 26^8 - 26^8 - 10^8 - 34^8 + 52^8 + 36^8 + 60^8 + 36^8 + 60^8 + 44^8 - 62^8 - 86^8 - 2(70^8) - 70(96^4 - 86^4 - 62^4 + 52^4) - 10(96^4 - 86^4 - 62^4 - 70^4 + 52^4 + 60^4 + 36^4 - 26^4).$$

Using a computational tool, we find this value to be 3257284313869120.

As you can see, I completely misjudged the difficulty of this problem when I looked at it. From now on, we will make sure to solve all of the problems before posing them!!!

Note: Due to the complexity and level of detail of this solution, I wrote a computer program that runs a modified brute force search through all possible passwords counting them. This program is included in a separate file called, alex.java. My computer program verifies this numerical answer.

3) Prove that every integer n > 6 can be expressed as the sum of two relatively prime integers that are both greater than 1.

Solution

First we must realize from Euclid's Algorithm that if d divides two integers a and b, then d must also divide their difference a – b.

Therefore, consecutive positive integers are always relatively prime since their difference will be 1.

Furthermore, if a and b are both odd with a difference of 2 or 4, then a and b are relatively prime, since the largest odd that divides into either 2 or 4 is 1. (Essentially, gcd(a, b) = gcd(a, |a - b|) = gcd(a, 2) = 1, or gcd(a, b) = gcd(a, |a - b|) = gcd(a, 4) = 1.)

So, for n > 6:

- If n is odd then n = 2k+1, where k +3 is an integer. Thus, n = k + (k + 1); which has a difference of 1.
- If n is even, then n = 2k, where k = 4 is an integer.
  - If k is even, n = (k – 1) + (k + 1); which has a difference of 2.
  - If k is odd, n = (k – 2) + (k + 2); which has a difference of 4.

4) Solve the following system of equations for x, y and z in terms of the constants a, b and c:

$$x + y = a \qquad x + z = b \qquad y + z = c$$

Solution

$$\begin{array}{rl} & x + y + 0 = a \\ + & x + 0 + z = b \\ & \underline{0 + y + z = c} \\ & 2x + 2y + 2z = a + b + c \longrightarrow x + y + z = \frac{a+b+c}{2} \end{array}$$

$$x + c = \frac{a+b+c}{2} \longrightarrow x = \frac{a+b+c}{2} - c$$

$$y + b = \frac{a+b+c}{2} \longrightarrow y = \frac{a+b+c}{2} - b$$

$$z + a = \frac{a+b+c}{2} \longrightarrow z = \frac{a+b+c}{2} - a$$

5) A palindromic number is one with a leading digit greater than 0 that is the same read forwards and backwards. How many different palindromic numbers of exactly 11 digits are divisible by 3?

<u>Solution</u>

) In an 11 digit palindrome we only need to focus on the first 6 digits since the last 5 digits are mirrored from the first 5 digits. There are 9 possibilities for the 1st digit since it can't be a 0, and 10 possibilities for digits 2-6. Digits 7-11 only have 1 possibility since they are mirrored.

___	___	___	___	___	___	___	___	___	___	___
9	10	10	10	10	10	1	1	1	1	1

Observations:

1) If the first 5 digits are divisible by 3, then the last 5 digits are also divisible by 3. Therefore, the middle digit, or sixth digit, must also be divisible by 3 (3,6, or 9) or be 0.
2) If the first 5 digits aren't divisible by 3, then the middle digit must be either (1,4,or 7), or (2, 5, or 8).

Therefore, we need to only look at the first 5 digits and determine if they are divisible by 3. There are 90,000 possible 5 digit numbers.

There are 90,000/3 = 30,000 five digit numbers divisible by 3.

30,000 * 4 possible middle digits = 120,000 possible palindromes with middle digit 0,3,6, or 9.

There are 90,000-30,000 = 60,000 five digit numbers not divisible by 3.

60,000 * 3 possible middle digits = 180,000 possible palindromes with middle digit (1,4, or 7), or (2, 5, or 8).

Therefore, there are 120,000 + 180,000 = 300,000 11-digit palindromes divisible by 3.

Investigations

1) A baseball team plays 45 games in 30 days. The team plays at least one game every day. Prove that there is some consecutive window of days in which the team plays exactly 14 games.

Solution

Define t_i to be the total number of games played from day 1 through day i, inclusive, for $1 \leq i \leq 30$. We know that $t_1 \geq 1$ and $t_{30} = 45$. Now, define $v_i = t_i + 14$, for $1 \leq i \leq 30$. Consider the set of 60 values $\{t_1, t_2, ..., t_{30}, v_1, v_2, ..., v_{30}\}$. The maximum value in the set must be $v_{30} = t_{30} + 14 = 59$, while we previously established that all the values in the set are positive. Thus, there are 60 different variables in the set that take on at most, 59 values. By the pigeonhole principle, this means that two of the variables listed in the set are equal.

First, note that $t_i \neq t_j$, for all $i \neq j$ because the team is guaranteed to play at least one game a day. This means that that total number of games played after a particular day is always strictly greater than all of the previous totals and strictly less than all of the future totals. Since the sequence v is constructed from t by shifting it by a constant value, it follows that that $v_i \neq v_j$, for all $i \neq j$. Thus, if there are to be two values in the set that are equal, we must have $t_j = v_i$, for some integers $i < j$.

This effectively means that on day j the total number of games played is equal to v_i, which is the total number of games played by day i, plus 14. Thus, by definition, we've found the existence of a window, from day i+1 through day j, inclusive, where exactly 14 games were played.

2) Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with all real roots. Prove that $2a^2 - 3b \geq 0$.

Solution

Let the roots of f(x) be r_1, r_2, r_3 and r_4. Then, $f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$. Expanding this out and equating coefficients for a and b, we find:

$$a = r_1 + r_2 + r_3 + r_4$$
$$b = r_1r_2 + r_1r_3 + r_14 + r_2r_3 + r_2r_4 + r_3r_4$$

$$2a^2 - 3b = (r_1 + r_2 + r_3 + r_4)^2 - 3(r_1r_2 + r_1r_3 + r_14 + r_2r_3 + r_2r_4 + r_3r_4)$$

$$= r_1^2 + r_2^2 + r_3^2 + r_4^2 + 2(r_1r_2 + r_1r_3 + r_14 + r_2r_3 + r_2r_4 + r_3r_4)$$
$$-3(r_1r_2 + r_1r_3 + r_14 + r_2r_3 + r_2r_4 + r_3r_4)$$

$$= r_1^2 + r_2^2 + r_3^2 + r_4^2 - (r_1r_2 + r_1r_3 + r_14 + r_2r_3 + r_2r_4 + r_3r_4)$$

$$= \frac{1}{2}(r_1 - r_2)^2 + \frac{1}{2}(r_1 - r_3)^2 + \frac{1}{2}(r_1 - r_4)^2 + \frac{1}{2}(r_2 - r_3)^2 + \frac{1}{2}(r_2 - r_4)^2 + \frac{1}{2}(r_3 - r_4)^2$$
$$\geq 0$$

Notice that in the second to last line we pair up the terms into perfect squares, realizing that if we factor out $\frac{1}{2}$ from each of these terms, we have enough of the perfect squares to match up with each of the product terms. Since each of the roots is real, we can ascertain that each of the terms in the sum on the second to last line is non-negative, allowing us to prove the assertion.

3) Imagine flipping a biased coin, with a probability of landing heads equal to $\frac{3}{4}$, 2n times. What is the probability that it will land heads an even number of times, in terms of n?

Solution

Consider the binomial expansion of $(\frac{3}{4}+\frac{1}{4})^{2n}$. Each term would represent the probability of heads landing a particular number of times. Specifically, the term $\binom{2n}{k}(\frac{3}{4})^k(\frac{1}{4})^{2n-k}$ represents the probability of flipping exactly k heads. The derivation for this result comes from the binomial distribution (http://www3.nd.edu/~rwilliam/stats1/x13.pdf, http://en.wikipedia.org/wiki/Binomial_distribution).

Thus, we want to sum each of the terms above where k is even: $\sum_{k=0}^{n}\binom{n}{2k}(\frac{3}{4})^k(\frac{1}{4})^{2n-2k}$.

In some sense, we want to cancel out the odd terms. Imagine the original binomial expression shown, but with a negative sign and consider both equations. We have:

$$(\frac{3}{4}+\frac{1}{4})^{2n}=\sum_{k=0}^{2n}\binom{n}{k}(\frac{3}{4})^k(\frac{1}{4})^{2n-k}$$

$$(\frac{3}{4}-\frac{1}{4})^{2n}=\sum_{k=0}^{2n}\binom{n}{k}(\frac{3}{4})^k(-\frac{1}{4})^{2n-k}$$

If we were to add these two equations, the left hand side is simply $1^{2n}+(\frac{1}{2})^{2n}$. On the right hand side, we see that some of the corresponding terms are identical and others are opposite. Specifically, whenever k is odd, $(-\frac{1}{4})^{2n-k}=-(\frac{1}{4})^{2n-k}$, which means that the corresponding terms, when added, cancel out and add to 0. When k is even, the two terms are identical. Thus, by adding the two right hand sides together, we get $2\sum_{k=0}^{n}\binom{n}{2k}(\frac{3}{4})^k(\frac{1}{4})^{2n-2k}$, which is precisely twice the sum we desire to obtain!!! It follows that our desired probability is:

$$\sum_{k=0}^{n}\binom{n}{2k}(\frac{3}{4})^k(\frac{1}{4})^{2n-2k}=\frac{1+(\frac{1}{2})^{2n}}{2}$$

4) Let a, b and c be positive integers. Let gcd(a, b, c) represent the greatest common divisor of a, b and c and lcm(a, b, c) be the least common multiple of a, b and c. Prove that

$$\frac{(\mathrm{gcd}(a,b,c))^2}{\mathrm{gcd}(a,b)\,\mathrm{gcd}(a,c)\,\mathrm{gcd}(b,c)} = \frac{(\mathrm{lcm}(a,b,c))^2}{lcm(a,b)lcm(a,c)lcm(b,c)}$$

Note: This problem was taken from the 1972 USA Mathematics Olympiad.

<u>Solution</u>

I have chosen it because I think it's a classical problem illustrating the Fundamental Theorem of Arithmetic. We can represent a, b and c in their unique prime factorizations, so let

$$a = \prod_{i=1}^{\infty} p_i^{\,a_i}, b = \prod_{i=1}^{\infty} p_i^{\,b_i}, c = \prod_{i=1}^{\infty} p_i^{\,c_i}$$

where p_i represents the i^{th} prime number. Recall that to find the gcd and lcm of a set of numbers already in their prime factorized forms, we simply find the minimum and maximum, respectively, of each corresponding exponent. Thus, we can evaluate the left hand side as follows:

$$\frac{(\mathrm{gcd}(a,b,c))^2}{\mathrm{gcd}(a,b)\,\mathrm{gcd}(a,c)\,\mathrm{gcd}(b,c)} = \frac{\left(\prod_{i=1}^{\infty} p_i^{\min(a_i,b_i,c_i)}\right)^2}{\left(\prod_{i=1}^{\infty} p_i^{\min(a_i,b_i)}\right)\left(\prod_{i=1}^{\infty} p_i^{\min(a_i,c_i)}\right)\left(\prod_{i=1}^{\infty} p_i^{\min(b_i,c_i)}\right)}$$
$$= \prod_{i=1}^{\infty} p_i^{\,2\min(a_i,b_i,c_i)-\min(a_i,b_i)-\min(a_i,c_i)-\min(b_i,c_i)}$$

Let median(x, y, z) denote the median value of the set x, y and z, which can also be thought of as the second smallest value. Notice that the minimum of all three corresponding exponents must be subtracted out twice from the exponent because it must appear exactly twice amongst the last three (negative) terms in the exponent to each prime. This means that the first term, 2min(a_i, b_i, c_i) completely cancels out with two of the subtracted terms for each i. This leaves a third term to be accounted for, that will always be part of one of the three last terms. That third term is the minimum amongst two terms which DON'T include the minimum of all three terms a_i, b_i, and c_i. Thus, we can rewrite the expression above as follows:

$$= \prod_{i=1}^{\infty} p_i^{\,-\mathrm{median}(a_i,b_i,c_i)}$$

Now, let's do a similar simplification on the right hand side:

$$\frac{(\mathrm{lcm}(a,b,c))^2}{\mathrm{lcm}(a,b)\,\mathrm{lcm}(a,c)\,\mathrm{lcm}(b,c)} = \frac{\left(\prod_{i=1}^{\infty} p_i^{\max(a_i,b_i,c_i)}\right)^2}{\left(\prod_{i=1}^{\infty} p_i^{\max(a_i,b_i)}\right)\left(\prod_{i=1}^{\infty} p_i^{\max(a_i,c_i)}\right)\left(\prod_{i=1}^{\infty} p_i^{\max(b_i,c_i)}\right)}$$
$$= \prod_{i=1}^{\infty} p_i^{\,2\max(a_i,b_i,c_i)-\max(a_i,b_i)-\max(a_i,c_i)-\max(b_i,c_i)}$$

The same logic we used about the minimum of two numbers and three numbers applies to the maximum. Namely, the maximum of all three values a_i, b_i and c_i must appear exactly twice in the last three terms, canceling out with the first term. Once again, what remains is a third term, which is the maximum of two terms, neither of which is the maximum of a_i, b_i and c_i. The maximum of this set is ALSO the median of the set. Thus we have that our expression above is equal to:

$$\prod_{i=1}^{\infty} {p_i}^{-\mathrm{median}(a_i, b_i, c_i)}$$

It follows that both the left-hand side and right-hand sides of the given equation are equal to one another.

5) Let the sets A = {z: $z^p = 1$} and B = {w: $w^q = 1$}, where gcd(p, q) = 1. Both A and B are sets of complex roots of unity. Prove that the set C = {zw : z ∈ A and w ∈ B} is a set of complex roots of unity of size pq.

Solution

Using DeMoivre's Theorem, the set A contains all terms of the form { $\cos\left(\frac{2\pi k}{p}\right) + isin(\frac{2\pi k}{p})$ | $k \in Z$ } and the set B contains all terms of the form { $\cos\left(\frac{2\pi m}{q}\right) + isin(\frac{2\pi m}{q})$ | $k \in Z$}. Thus, if we take a product of an arbitrary term in A and an arbitrary term in B, we get

$$(\cos\left(\frac{2\pi k}{p}\right) + isin(\frac{2\pi k}{p}))(\cos\left(\frac{2\pi m}{q}\right) + isin(\frac{2\pi m}{q})\ )$$

Once again, using DeMoivre's Theorem, we can express this product by simply adding the angles in question:

$$(\cos\left(2\pi(\frac{k}{p} + \frac{m}{q})\right) + isin(2\pi(\frac{k}{p} + \frac{m}{q}))$$

Simplifying, we get:

$$(\cos\left(2\pi(\frac{kq + mp}{pq})\right) + isin(2\pi(\frac{kq + mp}{pq}))$$

If we are to prove that the set C is the set of complex roots of unity of size pq, then we must show that the numerator in the fraction above can take on any integral value, but that it can't take on any non-integral values (which would correspond to values outside the set of roots of unity of size *pq*.) The latter part of the proof is easier because we know that *k*, *m*, *p*, and *q* are all integers, so their sums and/or products must be as well. Now we show that any integral numerator is achievable.

Since k and m can be any integers, each linear combination of p and q can be formed in the numerator. Specifically, the Extended Euclidean Algorithm shows that if gcd(p, q) = 1, there exist integers x and y such that px + qy = 1. If we wanted to create a product such that the angle in question was $\frac{2\pi n}{pq}$, where *n* is any integer, we can simply set *k* = *yn* and *m* = *xn*:

$$(\cos\left(2\pi(\frac{ynq+xnp}{pq})\right) + isin(2\pi(\frac{ynq+xnp}{pq})) = (\cos\left(2\pi(\frac{n(qy+xp)}{pq})\right) + isin(2\pi(\frac{n(qy+xp)}{pq}))$$

$$=(\cos\left(2\pi(\frac{n}{pq})\right) + isin(2\pi(\frac{n}{pq}))$$

This proves that the set C contains all of the roots of unity size *pq* and no other roots not in this set.