University of Central Florida
Online Mathematics Contest
Solutions: February 2014 (Year 1, Round 2)

Warm Ups

1) Let x be the length of the diagonal of a rectangle. If the ratio of the length to the width of the rectangle is 4, what is the perimeter of the rectangle in terms of x?

Solution

Let a be the width of one side of the rectangle and b be the length of another side.

$$a = 4b$$

Let the diagonal be represented by x which we can solve for with Pythagorean's Theorem

$$x^2 = a^2 + b^2$$

Now, we can substitute the value of a to solve for the length and width in terms of x

$$x^2 = (4b)^2 + b^2$$

$$x^2 = 17b^2$$

Solve for the positive value of b since the length cannot be negative.

$$b = \frac{x}{\sqrt{17}}$$

and since a = 4b

$$a = \frac{4x}{\sqrt{17}}$$

The perimeter is twice the width plus twice the width so the perimeter is

$$P = 2(a+b) = \frac{10x}{\sqrt{17}}$$

Note: One may rationalize the denominator if she chooses to do so.

2) Determine the solutions for t in the following inequality: $26 \leq 90 - e^t \leq 54$.

Solution

$26 \leq 90 - e^t \leq 54$	First, isolate the variable
$-64 \leq -e^t \leq -36$	Flip inequality after dividing by -1 and solve for t
$36 \leq e^t \leq 64$	
$\ln(36) \leq t \leq \ln(64)$	

So, t exists in the interval [ln(36), ln(64)].

3) Find a degree 4 (quartic) polynomial function f(x) such that f(1) = 0, f(3) = 0 and f(x) ≥ 0, for all real x.

Solution

Since the polynomial is always non-negative and we're given two points at which it's equal to zero, both of those points must be turning points and only one local maxima must exist in between x = 1 and x = 3. When x < 1, the function must be decreasing and when x > 3, it must be increasing. The only way the function can have two zeros that are also turning points is if those points are both repeated zeros. It follows that one polynomial that satisfies the conditions is $f(x) = (x-1)^2(x-3)^2$. Any constant multiple of this will work as well.

Another way of viewing this problem is using the two zeros and setting up variables for the coefficients of what remains. Thus, $f(x) = (x-1)(x-3)(x^2 + ax + b)$, for some a and b. Based on the restrictions, you find that $x^2 + ax + b > 0$ when x < 1, $x^2 + ax + b < 0$ when 1 < x < 3 and $x^2 + ax + b > 0$ when x > 3. Of course, this behavior is only consistent with this remaining quadratic having roots 1 and 3, so we arrive at the same conclusion.

4) The current in a river is flowing at 3 mph downstream. Without any current, Serena rows her kayak at 5 mph. She is planning a trip where she'll go downstream and then return to where she started, going back upstream. How far should she go downstream before turning around so that she returns to her starting spot exactly 2 hours after she left it?

Solution

With the current at her back she rows 8 mph while she only rows 2 mph on the way back, against the current. Let t be her time rowing downstream and t' be her time rowing back upstream. Since these distances are equal, we have

8t = 2t'
t' = 4t.

Thus, the time to row back is 4t. Since the total rowing time is 2 hours, we find that t + 4t = 2, so t = .4 hours. Thus, the total distance she'll travel downstream is simply (8 mph)(.4 hr) = 3.2 miles. It will take her 1.6 hours going at 2 mph to cover the distance on the way back.

5) Josiah is making hot chocolate for Jasmine from premade mixtures of milk and chocolate. Jasmine prefers her hot chocolate to have 6 parts milk for 1 part chocolate. Unfortunately, Josiah only has access to two mixtures of hot chocolate, neither of which is mixed to the ratio that Jasmine prefers. Mixture A has 3 parts milk for 1 part chocolate and Mixture B has 10 parts milk for 1 part chocolate. If Josiah wants to make Jasmine 8 ounces of hot chocolate according to her preference, how many ounces of mixture A should he use and how many ounces of mixture B should he use? (Note: The sum of your answers must be 8.)

<u>Solution</u>

Let Josiah use x ounces of mixture A and 8 – x ounces of mixture B. Since Jasmine wants a 6:1 ration of milk to chocolate, she needs a total of $8\left(\frac{1}{6+1}\right) = \frac{8}{7}$ ounces of chocolate total in her hot chocolate. Using the given information about mixtures A and B, we have:

$$x\left(\frac{1}{4}\right) + (8 - x)\left(\frac{1}{11}\right) = \frac{8}{7}$$
$$\frac{x}{4} + \frac{8 - x}{11} = \frac{8}{7}$$
$$\frac{11x}{44} + \frac{4(8 - x)}{44} = \frac{8}{7}$$
$$\frac{11x + 32 - 4x}{44} = \frac{8}{7}$$
$$\frac{7x + 32}{44} = \frac{8}{7}$$
$$7x + 32 = \frac{352}{7}$$
$$7x = \frac{128}{7}$$
$$x = \frac{128}{49} = 2\frac{30}{49}$$

Thus, Josiah must use $2\frac{30}{49}$ oz. of mixture A and $5\frac{19}{49}$ oz. of mixture B.

Exercises

1) Define a simple integer to be one that is divisible by 2, 3 or 5. How many simple integers are less than or equal to 1000000?

Solution

In general, there are $\left\lfloor \frac{n}{k} \right\rfloor$ values in the set {1, 2, …, n} that are divisible by k. This is simply because every kth value in the set is divisible by k, and in each group of k, it's the last element divisible by k, so the floor function naturally takes care of this border case.

From here, we just apply the Inclusion-Exclusion Principle for three sets to do our counting, noting that values that are divisible by a and b are simply the same as the number of values divisible by the least common multiple of a and b, for any positive integers a and b.

Answer = $\left\lfloor \frac{10^6}{2} \right\rfloor + \left\lfloor \frac{10^6}{3} \right\rfloor + \left\lfloor \frac{10^6}{5} \right\rfloor - \left\lfloor \frac{10^6}{6} \right\rfloor - \left\lfloor \frac{10^6}{10} \right\rfloor - \left\lfloor \frac{10^6}{15} \right\rfloor + \left\lfloor \frac{10^6}{30} \right\rfloor = 733334.$

2) What is the maximum integer k for which 1000! is divisible by 10^k?

Solution

We need to determine the exponent of 2 and 5 in the prime factorization of 1000! After some basic analysis, it'll be evident that the exponent to 2 is greater, so to answer our desired question, we only need to determine the exponent of 5 in the prime factorization of 1000!

Consider writing out the integers from 1 to 1000, in order, knowing that we're going to multiply them all: 1, 2, 3, 4, 5, …, 1000. We can quickly see that every 5th item will contain a factor of five. As in the previous question, a formula for now many times we see a factor of a prime p in the first n integers is simply $\left\lfloor \frac{n}{p} \right\rfloor$, for the same reasoning as in the previous problem. The issue of course becomes when we "cancel" a copy of five from 25 in the list, the leftover factor is itself a 5. In essence, once we cancel out every regular multiple of 5, we find that we didn't cancel all of the fives. In particular, all original multiples of 25 still have an extra five left. This reasoning continues recursively. Thus, in general, if we want to find the number of times a prime p divides evenly into n! we must take the following sum:

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

Of course, after the first few terms of the sum, the rest are zero. Effectively, we can stop computing terms in the sum once $p^i > n$. In our case, the last value of i that makes a contribution to the sum is I = 4, wince $5^4 = 635 < 1000$, but $5^5 = 3125 > 1000$. Adding, we get:

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 249$$

Thus the maximum value for k is k = 249. This means that there are exactly 249 zeroes at the end of 1000!

3) How many non-negative integer solutions are there to the equation a + b + c + d = 20, such that a < 10 and b > 5?

Solution

This question involves combinations with repetition. Since b > 5, we can go ahead let b = 6 + b', where b' >= 0. Thus, we are looking for the number of non-negative solutions to

a + b' + 6 + c + d = 20
a + b' + c + d = 14

with a < 10.

First, let's count all solutions without the restriction. Using the formula for combinations with repetition, we get a total of $\binom{14+3}{3} = \binom{17}{3}$ combinations of 14 items chosen from 4 distinct item types.

Now, we must subtract out of this total all solutions where a ≥ 10. We can simply do this by setting a = a' + 10 and solving for the number of equations to this adjusted equation:

a' + 10 + b' + c + d = 14
a' + b' + c + d = 4

There are a total of $\binom{4+3}{3} = \binom{7}{3}$ such solutions.

Thus, the answer to our original question is $\binom{17}{3} - \binom{7}{3} = 645$.

4) For all positive integers n, prove $2\left(\sqrt{n+1} - \sqrt{n}\right) < \frac{1}{\sqrt{n}}$.

Solution

Direct Proof:

$2\left(\sqrt{n+1} - \sqrt{n}\right) < \frac{1}{\sqrt{n}}$, where n is a positive number

$$2\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{2\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{n+1} + \sqrt{n}}$$

$$\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} * \frac{2}{2} = \frac{2}{2\sqrt{n}} = \frac{2}{\sqrt{n} + \sqrt{n}}$$

Therefore, $\frac{2}{\sqrt{n+1} + \sqrt{n}} < \frac{2}{\sqrt{n}+\sqrt{n}}$

This is true since you can clearly see the denominator is larger on the left side of the equation than the right side, thereby making the fraction smaller and less than the right side.

5) Let r and s be the roots of the quadratic function f(x) = $x^2 - 9x + 16$. Determine the quadratic function with leading coefficient 1 with roots r^3 and s^3.

<u>Solution</u>

Let *r* and *s* be the roots of the given equation. We can equate coefficients as follows:

$$x^2 - 9x + 16, \text{where r and s are roots}$$

$$x^2 - ax + b = (x - r)(x - s)$$

$$= x^2 - rx - sx + rs$$

$$= x^2 - (r + s)x + rs$$

$$(r + s) = 9 \qquad\qquad rs = 16$$

Now, let's use this information to obtain r^3s^3 and $r^3 + s^3$, the two desired coefficients in our solution:

$$9^3 = (r + s)^3$$

$$= r^3 + 3r^2s + 3rs^2 + s^3$$

$$= r^3 + 3rs(r + s) + s^3$$

$$729 = r^3 + s^3 + 3(16)(9)$$

$$r^3 + s^3 = 729 - 432 = 297$$

$$r^3s^3 = 16^3 = 4096$$

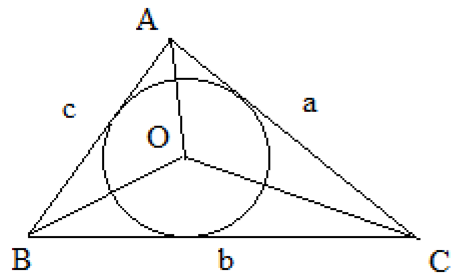
ANS: $\boldsymbol{f(x) = x^2 - 297x + 4096}$

Investigations

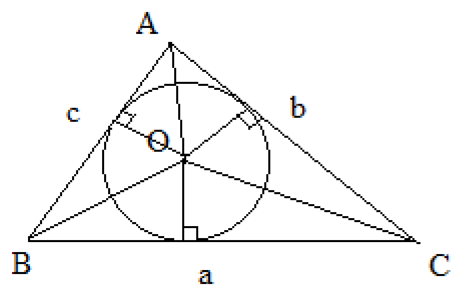
1) Prove, using basic geometric facts and Heron's formula for triangle area, prove that the inradius of a triangle with sides a, b and c, with $s = \frac{a+b+c}{2}$ is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

Solution

Heron's formula for an area of a triangle is, given sides a, b, and c, with s $=\frac{a+b+c}{2}$, is $\sqrt{s(s-a)(s-b)(s-c)}$. We can inscribe a circle in a triangle ABC and subdivide the original triangle into three triangles, AOB, BOC, AOC, where O is the incenter:



The area of the larger triangle is simply the sum of the areas of the three triangls AOB, BOC and AOC. To obtain these areas, draw the three altitudes of these triangles to the bases AB, BC and AC, respectively:



Each of these altitudes is equal to the inradius of the triangle. Let this value be r. Then, we can express the area of each of the triangles as follows:

$$Area(AOB) = \frac{1}{2}cr,\ Area(BOC) = \frac{1}{2}ar, \text{ and } Area(AOB) = \frac{1}{2}br$$

Noting that the area of the whole triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, we get the following:

$$\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br = r\left(\frac{a+b+c}{2}\right) = rs$$

Using this equation, we can solve for r in terms of the sides a, b and c of the triangle:

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

2) Determine the following infinite sum: $\sum_{i=0}^{\infty} i^2 p^i$, where 0 < p < 1, in terms of p.

<u>Solution</u>

We will use the pertubation method, which involves setting our sum to a variable S, multiplying it through by a value in the problem (in this case, p), and subtracting the two resulting sums.

$$S = \sum_{i=0}^{\infty} i^2 p^i = p + 4p^2 + 9p^3 + 16p^4 + \cdots$$

$$pS = \sum_{i=0}^{\infty} i^2 p^{i+1} = 0p + p^2 + 4p^3 + 9p^4 + 16p^5 + \cdots$$

Subtracting these two equations, lining up like terms, we get:

$$S - pS = \sum_{i=0}^{\infty} (2i+1)p^{i+1} = p + 3p^2 + 5p^3 + 7p^4 + \cdots$$

Now, let *T* = *S* - *pS* = *S(1 - p)*. Let's use this same subtraction method to solve for T:

$$T = \sum_{i=0}^{\infty} (2i+1)p^{i+1} = p + 3p^2 + 5p^3 + 7p^4 + \cdots$$

$$pT = \sum_{i=0}^{\infty} (2i+1)p^{i+2} = p^2 + 3p^3 + 5p^4 + \cdots$$

Subtracting again, we get:

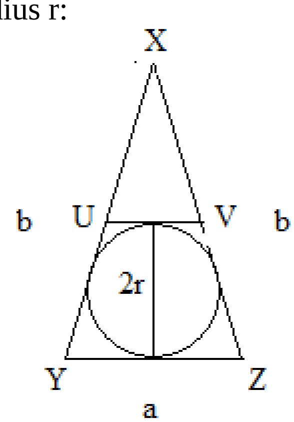
$$T - pT = p + \sum_{i=2}^{\infty} 2p^i = p + 2p^2 \sum_{i=0}^{\infty} p^i = p + \frac{2p^2}{1-p} = \frac{p(1-p) + 2p^2}{1-p} = \frac{p(1+p)}{1-p}$$

Finally, solving, we find T = $\frac{p(1+p)}{(1-p)^2}$ and S = $\frac{p(1+p)}{(1-p)^3}$.

3) Let T be a isosceles triangle with a base length of a, and two sides of length b. Let C_1 be the incircle of T. Let C_i be the circle inscribed between the two sides of length b and C_{i-1}, for all i > 1. Determine $\sum_{i=1}^{\infty} Area(C_i)$.

<u>Solution</u>

We can reuse our work from Investigation #1 and utilize similar triangles. Here is a picture with a single inscribed circle with radius r:



The semiperimeter of this triangle is $s = b + \frac{a}{2}$. Thus, using the result derived in Investigation #1, we find that $r = \sqrt{\frac{(s-a)(s-b)(s-b)}{s}} = \sqrt{\frac{(b-\frac{a}{2})(\frac{a}{2})(\frac{a}{2})}{b+\frac{a}{2}}} = \frac{a}{2}\sqrt{\frac{2b-a}{2b+a}}$, and $d = 2r = a\sqrt{\frac{2b-a}{2b+a}}$.

Since the smaller triangle shown is similar to the larger one, and C_2 is simply the inscribed circle in this smaller triangle, we find that the areas of each circle are in geometric ratio to one another, as are the radii. We can uncover the common ratio of the radii by determining the ratio of the heights of the two triangles XUV and YYZ:

The height of XYZ to base XY is $\sqrt{b^2 - ({}^{a}/_{2})^2} = \frac{1}{2}\sqrt{(2b-a)(2b+a)}$.

The height of XUV to base UV is $\frac{1}{2}\sqrt{(2b-a)(2b+a)} - a\sqrt{\frac{2b-a}{2b+a}} = \sqrt{\frac{2b-a}{2b+a}}(b - \frac{a}{2})$.

The ratio of the latter to the former is $\frac{\sqrt{\frac{2b-a}{2b+a}}(b-\frac{a}{2})}{\frac{1}{2}\sqrt{(2b-a)(2b+a)}} = \frac{\frac{1}{2}\sqrt{\frac{2b-a}{2b+a}}(2b-a)}{\frac{1}{2}\sqrt{(2b-a)(2b+a)}} = \frac{2b-a}{2b+a}$.

The area of the first circle is $\pi(\frac{a}{2}\sqrt{\frac{2b-a}{2b+a}})^2 = \frac{\pi a^2(2b-a)}{4(2b+a)}$. If the ratios of successive radii are $\frac{2b-a}{2b+a}$, then the ratio of these corresponding areas is $(\frac{2b-a}{2b+a})^2$. Thus, our final sum is

$$\frac{Area(C_1)}{1 - AreaRatio} = \frac{\frac{\pi a^2(2b-a)}{4(2b+a)}}{1 - (\frac{2b-a}{2b+a})^2} = \frac{\pi a(2b-a)(2b+a)}{32b}$$

4) A dice game is played as follows: Roll a pair of fair six-sided dice. If the sum of the dice rolled is 12, you win. If not, roll the pair again. If this total is less than or equal to the previous total rolled, you lose. Continue rolling until either you lose because your next roll is less than or equal to the previous roll, or your total is 12, in which case you win. Given that Samantha won the game, what is the expected number of times she rolled the dice?

Solution

Consider all sequences of rolls that lead to winning the game. We know the last roll must be 12. All the previous rolls must be a subsequence of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. There are 2^{10} such subsequences. The probability of each subsequence is simply the product of getting each roll, since the order of the rolls is fixed.

Let p_i be the probability of rolling a sum of i on two dice and Let the set $S = \{p_i | 2 \leq i \leq 11\}$. Let's calculate the probability of winning the game. It's just $p_{12} \prod_{i=2}^{11}(1 + p_i)$. Note that when we multiply out this product, we will have every term that is a product of some subset of S ={p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}}, and each subset of this set maps to a unique sequence of rolls in ascending order before rolling a 12. We multiply since each roll is independent of the others.

$$p_{12} \prod_{i=2}^{11}(1 + p_i) = \frac{1}{36} \times \frac{37}{36} \times \frac{38}{36} \times \frac{39}{36} \times \frac{40}{36} \times \frac{41}{36} \times \frac{42}{36} \times \frac{41}{36} \times \frac{40}{36} \times \frac{39}{36} \times \frac{38}{36} \sim .0679441217.$$

I haven't found a strictly mathematical way to approach the question posed. Instead, I've written two programs: (a) FebGame.java - which calculates the desired sum corresponding to the answer to the question, and (b) FebGameSim.java - which simulates playing the game 1000000 times and calculates the experimental value of this expectation. After running the latter program multiple times, it gives an answer within .01 of the theoretical expected value calculated in the first program. This theoretical value is roughly 1.872895921.

We derive the equation to determine this value as follows:

We must sum over all possible ways of winning the game. For any subset T of S previously defined, define f(T) = $p_{12} \prod_{p_i \in T} p_i$. For example, if T = {$p_2$, p_8, p_{10}}, then f(T) = $\frac{1}{36} \times \frac{5}{36} \times \frac{3}{36} \times \frac{1}{36} = \frac{15}{36^3}$, corresponding to the probability of rolling a 2, followed by an 8, followed by a 10, followed by a 12. The contribution to expectation of the number of turns is this probability multiplied by the size of the set T, plus 1. Finally, to restrict our sample space to wins, we must divide this probability by the probability of winning. Thus, mathematically, an expression equal to the answer of the question posed is as follows:

$$\frac{\sum_{T \subseteq S}(|T| + 1)f(T)}{p_{12} \prod_{i=2}^{11}(1 + p_i)}$$

5) Let n be a positive integer. Determine $\sum_{i=1}^{n}\left[\left(\cos\left(\frac{2\pi i}{2n+1}\right)-1\right)\left(2\cos\left(\frac{2\pi i}{2n+1}\right)+1\right)\right]$.

Solution

Let $\theta=\frac{2\pi}{2n+1}$ and manipulate the sum so it's a bit more simple:

$$\sum_{i=1}^{n}[(\cos(i\theta)-1)(2\cos(i\theta)+1)]=$$
$$\sum_{i=1}^{n}[2cos^2(i\theta)-\cos(i\theta)-1]=$$
$$\sum_{i=1}^{n}(2cos^2(i\theta)-1)-\sum_{i=1}^{n}\cos(i\theta)=$$
$$\sum_{i=1}^{n}cos(2i\theta)-\sum_{i=1}^{n}\cos(i\theta)$$

Both sums represent adding up ***some*** of the real parts of the roots of unity of 2θ and θ, respectively. The second summation is easier to handle, so let's look at that first. Note that

$$0=\sum_{i=0}^{2n}\cos(i\theta)=1+\sum_{i=1}^{n}\cos(i\theta)+\sum_{i=n+1}^{2n}\cos(i\theta)$$

because the initial sum from 0 to 2n adds the real parts of all the $(2n+1)^{st}$ roots of unity. Secondly, note that the terms in the second sum from i = n+1 to 2n have the ***exact same values*** as the terms in the first sum from i = 1 to n, because cos(π+θ) = cos(π – θ), and the terms are symmetrical about π. Let the value of this sum be X, we then get

$$0=1+X+X$$
$$X=-\frac{1}{2}$$

Now, we aim to show that the first sum has the same exact terms as the second sum and also equals $-\frac{1}{2}$. We do this as follows. Certainly, some of the terms in the first sum that correspond to angles less than π must appear in the second sum, since the angles in the first sum are multiples (double) of angles in the second sum. Now, let's consider each of the terms in the first sum that correspond to angles in between π and 2π. First, note that none of these terms are equal, as cosine is an increasing function in this interval. Second, note that none of these terms may be equal to the terms corresponding to the angles in between 0 and π, since if we were to continue the sum, it would end up hitting all the real parts of the unique $(2n+1)^{st}$ roots of unity. Thus, if we can show that the first term with an angle in between π and 2π maps to an equivalent term in between 0 and π from the first sum, and noting that the spacing between angles is equal, we will have shown that there's a one-to-one correspondence between the terms in the two summations. From there, we can conclude that the sum of the first sum is $-\frac{1}{2}$ and our final result for the posed question is 0, as well.

Consider a term of the form $\frac{4\pi i}{2n+1}$, for i, $\left\lfloor\frac{n}{2}\right\rfloor$ < i ≤ n, which maps to an angle in between π and 2π. If n is odd, the smallest value of i that satisfies the restriction is i = $\frac{n+1}{2}$. The corresponding minimal angle in the range is $\frac{4\pi(n+1)/2}{2n+1} = \frac{(2n+2)\pi}{(2n+1)}$. We know that

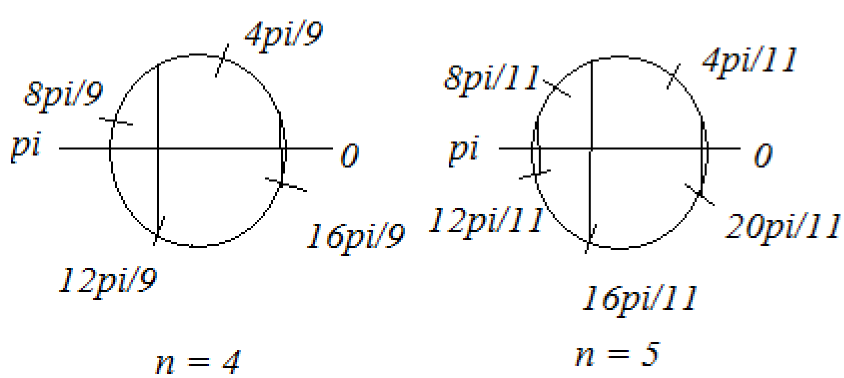
$$\cos\left(\frac{(2n+2)\pi}{(2n+1)}\right) = \cos\left(\frac{2n\pi}{(2n+1}\right)$$

from the previous cosine identity. Note that this previous angle wasn't part of the earlier part of the sum because the prior term was $\cos\left(\frac{(2n-2)\pi}{(2n+1)}\right)$. Thus, this first angle maps to a value from the latter sum that wasn't yet listed in this sum. Inductively, based on the spacing of the angles, we can show that the rest of the angles in the sum map to terms from the latter summation.

Now, the only case we have left to consider is even n. Then, the first term will satisfy i = $\frac{n}{2}+1$. Plugging in we get the angle $\frac{4\pi(n/2+1)}{2n+1} = \frac{(2n+4)\pi}{(2n+1)}$. Using our identity again, we have

$$\cos\left(\frac{(2n+4)\pi}{(2n+1)}\right) = \cos\left(\frac{(2n-2)\pi}{(2n+1}\right)$$

Once again, this term wasn't already part of our sum because the previous term was $\cos\left(\frac{(2n)\pi}{(2n+1)}\right)$. The argument follows similarly as the previous one. Visually, consider the x-coordinates of the points on the unit circle induced by the first sum when n = 4 and n = 5:



In the first example, we see that the angle $\frac{12\pi}{9}$ has the same x-coordinate/cosine as the angle $\frac{6\pi}{9}$, which IS a term in the sum $\sum_{i=1}^{n}\cos(i\theta)$, for n = 4, not yet seen in $\sum_{i=1}^{n}\cos(2i\theta)$. Similarly $\frac{16\pi}{9}$

has the same x-coordinate/cosine as the angle $\frac{2\pi}{9}$, also a term in $\sum_{i=1}^{n}\cos(i\theta)$, for n = 4, not yet seen in $\sum_{i=1}^{n}\cos(2i\theta)$.

In the second example, we have $\cos\left(\frac{12\pi}{11}\right) = \cos\left(\frac{10\pi}{11}\right)$, where the latter term is included in $\sum_{i=1}^{n}\cos(i\theta)$, for n = 5. The other equalities are $\cos\left(\frac{16\pi}{11}\right) = \cos\left(\frac{6\pi}{11}\right)$ and $\cos\left(\frac{20\pi}{11}\right) = \cos\left(\frac{2\pi}{11}\right)$. Using these equalities, we can see that $\sum_{i=1}^{n}\cos(2i\theta)$, for n = 5 is equal to

$$\cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{2\pi}{11}\right)$$

which is identical to that $\sum_{i=1}^{n}\cos(i\theta)$, for n = 5, but just adds the terms in a different order.