University of Central Florida
Online Mathematics Contest
Solutions: February 2015 (Year 2, Round 3)

Warm Ups

1) The ratio of the shorter side of a rectangle to its longer side is equal to the ratio of the longer side of the rectangle to the sum of the shorter and longer side. If the area of the rectangle is $2+2\sqrt{5}$, what are the lengths of the sides of the rectangle?

Solution
Let the shorter side of the rectangle be length x with x > 0, and the longer side be length rx, where r is the ratio in question. We then have the following equation based on the ratio r:

$$x + rx = r^2 x$$
$$x(r^2 - r - 1) = 0$$

Since we know that x > 0, it follows that $(r^2 - r - 1) = 0$. Solving the quadratic, we find, $r = \frac{1\pm\sqrt{5}}{2}$. Since we know that $r > 0$, we must have that $r = \frac{1+\sqrt{5}}{2}$. The area of the rectangle is rx^2. This gives us the equation:

$$\left(\frac{1+\sqrt{5}}{2}\right)x^2 = 2 + 2\sqrt{5}$$
$$x^2 = \frac{2(2+2\sqrt{5})}{(1+\sqrt{5})} = \frac{4(1+\sqrt{5})}{(1+\sqrt{5})} = 4$$

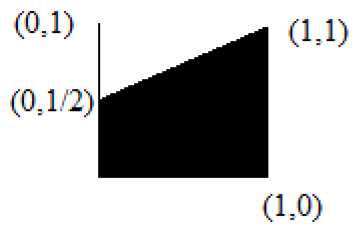
Since x > 0, it follows that x = 2 and the sides of the rectangle are 2 and $1+\sqrt{5}$.

2) We select a real number, x, at random from the range (0, 1). Then we select a real number, y, at random from the range (x, 1). What is the expected value of y?

Solution

For a given value x, y is uniformly chosen from (x, 1). This means that the expected value for y is the average of the two endpoints of the range, $\frac{1+x}{2}$. Now, we desire to find the average value of this function as x ranges from 0 to 1, exclusive. In the Cartesian plane, we seek to graph the line above, $y = \frac{x+1}{2}$, with x in (0 , 1) and find the average value of y. Since this is a linear function, the average value is obtained when $x = \frac{1}{2}$. Thus, the desired average value for y is $\frac{1/2+1}{2} = \frac{3}{4}$.

Alternatively, we can find the area of the following figure:



Since the length of the base is 1, the area of the figure represents the average value of y over the span. If x had a different range, we'd simply divide the area of the corresponding figure by the size of the range of x. The area of the figure is also $\frac{3}{4}$, which we can obtain using the formula for the area of a trapezoid.

3) An item is repeatedly discounted by 10%. How many times does it have to be discounted before the effective discount from the original price is at least 90%? (Note: derive an expression to your answer without a calculator. You may you the calculation as your final step, to find the numerical value of your expression.)

Solution

If we discount an item with an original cost of C n times with a discount of 10% each time, our price will be $(.9)^n C$. If we want an effective discount of at least 90%, we want to find the smallest positive integer n, such that

$$(.9)^n C < .1C$$
$$(.9)^n < .1$$
$$\ln((.9)^n) < \ln(.1)$$
$$n\ln(.9) < \ln(.1)$$
$$n > \frac{\ln(.1)}{\ln(.9)} \sim 21.85$$

It follows that we must apply the discount 22 times before the effective discount is 90% or more of the original price.

4) Shemina flips a fair coin n times in a row. What is the probability that she never sees the same outcome on two consecutive tosses?

Solution

There are a possible 2^n sequences of tosses she could make. Of these, there are only 2 that don't have the same outcome on consecutive tosses at some point. Namely, the first toss can be either heads or tails, but after that, each subsequent toss must be the opposite of the previous toss. Thus, our desired probability is $\frac{2}{2^n} = \frac{1}{2^{n-1}}$.

5) The sum of five terms of an arithmetic sequence is 50 and the product of those five terms is 58240. What are the five terms of the sequence?

Solution

Let x be the middle term of the sequence and d be the common difference. We then obtain the two following equations:

$$(x-2d)+(x-d)+x+(x+d)+(x+2d)=50$$
$$(x-2d)(x-d)x(x+d)(x+2d)=58240$$

Using the first equation, we find $5x=50$, so $x=10$. Now, substitute this into the other equation to yield:

$$(10-2d)(10-d)10(10+d)(10+2d)=58240$$
$$(100-4d^2)(100-d^2)=5824$$
$$10000-500d^2+4d^4=5824$$
$$4d^4-500d^2-4176=0$$
$$d^4-125d^2-1044=0$$
$$(d^2-9)(d^2-116)=0$$
$$d=\pm 3, \pm 2\sqrt{29}$$

Thus, there are four possible sequences that satisfy the given information:

$$4,7,10,13,16$$
$$16,13,10,7,4$$
$$10-4\sqrt{29}, 10-2\sqrt{29}, 10, 10+2\sqrt{29}, 10+4\sqrt{29}$$
$$10+4\sqrt{29}, 10+2\sqrt{29}, 10, 10-2\sqrt{29}, 10-4\sqrt{29}$$

Exercises

1) Students are sitting for an exam in desks arranged in one long row. There are a total of 17 desks and 5 students taking the exam. The professor would like the students arranged such that there is at least one empty desk in between any pair of students. How many different sets of desks can the students sit in that satisfy this constraint?

Solution
Let x_i be the number of empty desks to the left of the i^{th} student for $1 \leq i \leq 5$, and let x_6 be the number of empty desks to the right of the fifth student. We have that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 17 - 5$$

where all of the x_i's are integers, satisfying $x_1, x_6 \geq 0$ and $x_2, x_3, x_4, x_5 \geq 1$. Since we know that the middle four terms are 1 or greater, subtract 1 from each of these, changing their restriction to be non-negative, and compensate by subtracting 4 from the right hand side of the equation as well:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 8$$

We seek the number of non-negative integer solutions to the equation above. This is exactly a combination with repetition for choosing 8 items out of 6 distinct ones. This can be done is $\binom{8+6-1}{6-1} = \binom{13}{5}$ ways.

2) Prove that the system of equations below has no solutions for positive real values for a, b and c:

$$a^2 + b^2 = c^2$$
$$\sqrt{a} + \sqrt{b} = \sqrt{c}$$

Solution
Square the second equation to yield: $a + 2\sqrt{ab} + b = c$. Solving for $a+b$, we get:

$$a + b = c - 2\sqrt{ab}$$

Now, square both sides of this equation:

$$a^2 + 2ab + b^2 = c^2 - 4c\sqrt{ab} + 2ab$$
$$a^2 + b^2 = c^2 - 4c\sqrt{ab}$$

Since a, b, and c are all positive, it follows that $a^2 + b^2 = c^2 - 4c\sqrt{ab} < c^2$, proving that the given system of equations has no solutions in positive real numbers.

3) Find all positive integer solutions (x, y, z, w) with x ≤ y ≤ z ≤ w such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=1$.

Solution

Note that x > 1, because if x = 1, the left hand side would exceed 1 and that x ≤ 4, because if x = 4, then the left hand side can be at most 1. In fact, we can quickly see that when x = 4, the only solution is (4, 4, 4, 4).

Thus, we must find all solutions where x = 2 and x = 3.

Start with x = 2: $\frac{1}{2}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=1$, so $\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{2}$. Using the same type of logic, bounding y by x from below and noting that $\frac{3}{y}\geq\frac{1}{2}$, we find that $2<y\leq 6$. When y = 6, we have the solution, (2, 6, 6, 6). Now, we must find all solutions for y = 3, y = 4 and y = 5. Plug in y = 3 and simplify to get $\frac{1}{z}+\frac{1}{w}=\frac{1}{6}$. We can obtain all solutions with a brute force search for z, noting that $6<z\ \leq 12$. These solutions are (2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15) and (2, 3, 12, 12). Next, plug in y = 4 and simplify to get $\frac{1}{z}+\frac{1}{w}=\frac{1}{4}$. This yields the solutions (2, 4, 5, 20), (2, 4, 6, 12), and (2, 4, 8, 8). Finally, plug in y = 5 and simplify to get $\frac{1}{z}+\frac{1}{w}=\frac{3}{10}$. This yields the solution (2, 5, 5, 10).

Now, plug in x = 3 and simplify to get $\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{2}{3}$. We must have y = 3 or y = 4, since $\frac{3}{5}<\frac{2}{3}$. Plug in y = 3 and simplify to get $\frac{1}{z}+\frac{1}{w}=\frac{1}{3}$. The corresponding solutions are (3, 3, 4, 12), and (3, 3, 6, 6). Plug in y = 4 and simplify to get $\frac{1}{z}+\frac{1}{w}=\frac{5}{12}$. This has one solution, (3, 4, 4, 6).

Thus, all of our solutions in lexicographical order are:

(2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 3, 12, 12) (2, 4, 5, 20), (2, 4, 6, 12), (2, 4, 8, 8), (2, 5, 5, 10), (2, 6, 6, 6), (3, 3, 4, 12), (3, 3, 6, 6), (3, 4, 4, 6) and (4, 4, 4, 4).

4) A three digit integer *x* is added to a four digit integer *y* and no carrying is required. How many possible values are there for the ordered pairs (*x*, *y*)?

Solution

The least significant digits must add to 9 or less. Let these digits be x_0 and y_0, respectively. We want to find the number of non-negative integer solutions to $x_0+y_0\leq 9$. Add a slack variable, s_0, to the left hand side of the equation to be equal to the difference between 9 and the sum x_0+y_0. Now, we want the number of non-negative solutions to $x_0+y_0+s_0=9$. Using combinations with repetition, we find the number of solutions to be $\binom{9+3-1}{3-1}=\binom{11}{2}=55$. There are also *55* ways to set the tens digits of the two numbers. Let x_2 be the hundred's digit of x. Since x has only three digits, we see that $x_2\geq 1$. Thus, in our set up in writing out the equations above, we can subtract 1 from both sides, resetting x_2 with the restriction $x_2\geq 0$ and the equation

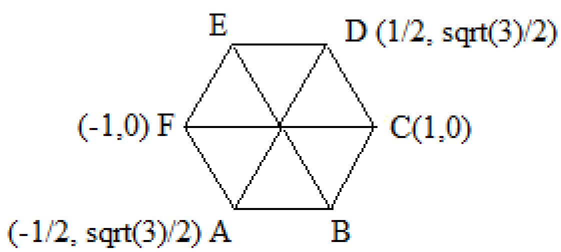
$x_2 + y_2 + s_2 = 8$. This has $\binom{10}{2} = 45$ possible solutions. Finally, the thousand's digit of y can be one of 9 values. Thus, the total number of ordered pairs that satisfy the query is $9 \times 45 \times 55 \times 55 = 1225125$.

5) Find the area of the largest regular hexagon that fits inside a square of side length 1.

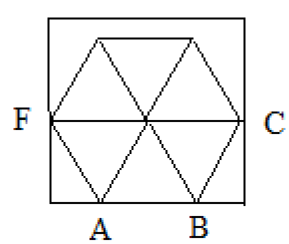
Solution

Let's start with a regular hexagon with side length 1 and solve for the smallest square it will fit inside. Then, we can linearly scale our solution to solve the given problem.

Put the hexagon, ABCDEF, in the Cartesian plane, centered at (0, 0), with sides parallel to the x-axis:



Note that if we were to put a square around this hexagon with the sides parallel to the x-axis and y-axis, the smallest such square that would satisfy the requirements would be size 2, since FC is parallel to the x-axis and has length 2. Also, note that in this case, both sides ED and AB would NOT be touching the square. Namely, the distance between ED and AB is $\sqrt{3}$, which is strictly less than two. So, our picture looks like this, roughly:



Imagine centering the square at the origin. so neither AB nor ED touched its bottom or top, respectively. Then, imagine rotating the hexagon about the origin, counter-clockwise. In doing so, the hexagon, would NOT touch the square at all, meaning that we could further reduce the size of the square. The slope of line segment AD is $\sqrt{3}$ in the current picture. This line forms a 60º angle with the positive x-axis. As we do our rotation, the angle that the line FC forms with the positive x-axis increases also and the difference in x-coordinates between F and C decreases. Of course, if we rotate 30º, then AD will be vertical and we'll still require a square of size 2 with sides parallel to the x-axis and y-axis. Thus, we wish to rotate an angle of θ, where 0º < θ < 30º.

Our rotation must be such that the difference in x-coordinates between F and C is the ***SAME*** as the difference in y-coordinates between A and D. Intuitively, we must catch the "cross-over point" where as the height of the hexagon is increasing (as measured in difference of y-coordinates of A an D) and the width of the hexagon is decreasing (as measured in difference of x-coordinates of F and C), where these two values match.

If we rotate an angle of θ, then the line AD will form an angle of (θ+60º) with the positive x-axis and FC will form an angle of θ. The height of the hexagon, as previously defined will be $2\sin(\theta + 60°)$ and the width of the hexagon, as previously defined will be $2\cos(\theta)$. (We can obtain these by looking at the right triangles formed by using AD and FC as the hypotenuse and using sides parallel to the x-axis and y-axis.)

Set these two equal to each other, as specified:

$$2\sin(\theta + 60°) = 2\cos(\theta)$$
$$\sin(\theta + 60°) = \cos(\theta)$$
$$\sin(\theta + 60°) = \sin(90° - \theta)$$

Thus, we have either $\theta + 60° = 90° - \theta + 360n$, for $n \in Z$, or
$\theta + 60° = 180° - (90° - \theta) + 360n$, for $n \in Z$.

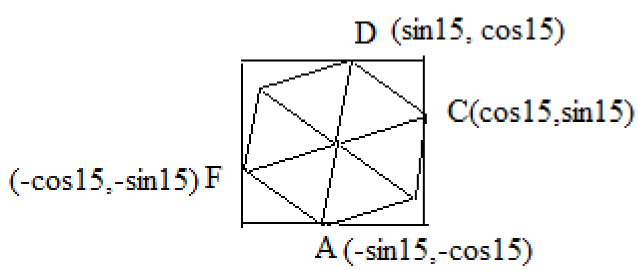
The first option will lead us to a solution in our desired range:

$$\theta + 60° = 90° - \theta$$
$$2\theta = 30°$$
$$\theta = 15°$$

It follows that our minimum-sized bounding square of a regular hexagon with side 1 is be $2\cos(15°)$. Scaling this figure down, we multiply each side by $\frac{1}{2\cos(15°)}$, thus our hexagon has side length $\frac{1}{2}\sec(15°)$. Using the subtraction angle identity for cosine, we find:

$$\cos(45° - 30°) = cos45°cos30° + sin45°sin30° = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

Using this, we get our side length to be $\frac{1}{2} \times \frac{4}{\sqrt{2}+\sqrt{6}} = \frac{2}{\sqrt{6}+\sqrt{2}} \times \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{2(\sqrt{6}-\sqrt{2})}{6-2} = \frac{\sqrt{6}-\sqrt{2}}{2}$. Using that cos 15º = sin 75º and sin 15º = cos 75º, we get our final picture as follows:



Investigations

1) a. Show that 2015 is a difference of two cubes.

b. What is the next year that is a sum or difference of two cubes?

c. What was the most recent previous year that was a sum or difference of two cubes?

Solution

a) Let x and y be positive integers with

$$x^3 - y^3 = 2015$$
$$(x-y)(x^2+xy+y^2) = 5 \times 13 \times 31$$

To systematically search for a solution, let $x - y$ equal the different factors of 2015, realizing that it must equal a factor less than the square root of 2015. First try $x - y = 5$. As y increases, x does as well. Plugging in $y = 8$ yields $x = 13$ and $x^2 + xy + y^2 = 337$. This value is too small since in order for the equation to work out it must be 403. Thus, we know that y must be larger. Try $y = 9$ yields $x = 14$ and $x^2 + xy + y^2 = 403$, as desired. Thus, we find that $14^3 - 9^3 = 2015$.

b) First let's quickly get the optimal answer for sums. There are only 12 cubes less than the current year: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331 and 1728. If we attempt to add two of these to get close to 2015, the larger of the two must either be 1331 or 1728, since 2 x 1000 = 2000 < 2015. The minimum solution formed with 1331 is 1331 + 729 = 2060. The minimum solution formed with 1728 is 1728 + 343 = 2071. Thus, for sums, the next year that is the sum of two cubes is 2060.

In considering differences, note that if we fix x - y, as y increases, the difference of cubes $x^3 - y^3$ increases. Thus, for any fixed difference between x and y, we simply seek the minimum y that allows for $x^3 - y^3$ above a particular threshold.

Plug in successive values for x, starting at x = 13. We know that we like $x^3 - 2015$ to be slightly larger than an actual cube. Here is a chart of the first few of those values:

x	$x^3 - 2015$	Nearest Smaller Cube	Result
13	182	125	$13^3 - 5^3 = 2072$
15	1360	1331	$15^3 - 11^3 = 2044$
16	2081	1728	$16^3 - 12^3 = 2368$
17	2898	2744	$17^3 - 14^3 = 2169$

At this point, our search is restricted to situations where $x - y = 2$ and $x - y = 1$. Let's consider each. In the first case we have:

$$x^3 - y^3 = (y+2)^3 - y^3 = 6y^2 + 12y + 8$$

Setting $6y^2 + 12y + 8 > 2015$, we get the inequality $6y^2 + 12y - 2007 > 0$. Solving for the positive root of the quadratic for y, we get $y = \frac{-12+\sqrt{144+4(6)(2007)}}{2(6)} \sim 17.31$. Thus, our best solution with $x - y = 2$ is $20^3 - 18^3 = 2168$.

Finally, repeat this process for $x - y = 1$. In this case we have

$$x^3 - y^3 = (y+1)^3 - y^3 = 3y^2 + 3y + 1$$

Setting $3y^2 + 3y + 1 > 2015$, we find $3y^2 + 3y - 2014 > 0$. Solving the quadratic for q, we get: $y = \frac{-3+\sqrt{9+4(3)(2014)}}{2(3)} \sim 25.41$. It follows that for this case, the best solution is $27^3 - 26^3 = 2107$.

Thus, the next year that is either a sum or difference of cubes is: $15^3 - 11^3 = 2044$.

c) Looking back at the previous analysis, we can essentially repeat it. Looking at the sums, we find our best answer with 1000 is 1000 + 1000 = 2000. With 1331, our best answer is 1331 + 512 = 1843 and our best answer with 1728 is 1728 + 216 = 1944.

For differences, create the same table for the first few cubes:

x	$x^3 - 2015$	Nearest Larger Cube	Result
13	182	216	$13^3 - 6^3 = 1981$
14	729	1000	$14^3 - 10^3 = 1744$
15	1360	1728	$15^3 - 12^3 = 1647$
16	2081	2197	$16^3 - 13^3 = 1899$

Now, we can just do the analysis for where $x - y = 2$ and $x - y = 1$. Luckily, we already solved the relevant quadratics, so we just have to check $19^3 - 17^3 = 1946$ and $26^3 - 25^3 = 1951$.

Thus, the last year that was either a sum or difference of cubes was the year 2000, since $10^3 + 10^3 = 2000$

2) Suppose that $a_1, a_2, \dots, a_n$ is a permutation of 1, 2, ... , n .

a. What's the smallest possible value of $|a_1 - a_2| + |a_2 - a_3| + \cdots + |a_{n-1} - a_n| + |a_n - a_1|$?

b. How many permutations achieve this minimum value?

<u>Solution</u>

a) What's the smallest possible value of $|a_1 - a_2| + |a_2 - a_3| + \cdots + |a_{n-1} - a_n| + |a_n - a_1|$?

Imagine a sequence $a_1, a_2, \dots a_n$ as movements on the x-axis by some particle. So, the particle starts at $x = a_1$ and then moves to $x = a_2$, and so forth. The sum designated above is simply how far the particle moves in n time steps if it's to return back to where it started after visiting each unique integer point from 1 to n, inclusive.

We know that at some time the particle must be at $x = 1$ and at another time it must be at $x = n$. Since its movement is cyclic, we can calculate how far it moves by using any item on the list as the starting point and simply following the sequence in cyclic order. (For example, if our sequence is 2, 4, 1, 5, 3, we can obtain the correct sum it generates by looking at the sequence 1, 5, 3, 2, 4, since the latter is a left cyclic shift of the former by 2 terms.) Take any sequence and shift it cyclically to an equivalent sequence that starts at 1. We know that at some point our particle must move from at $x = 1$ to $x = n$. At a minimum, this requires a movement of $n - 1$. Then, to complete our cycle, we must go from $x = n$ back to $x = 1$ by the very last move in our sequence. This is another movement of $n - 1$. Thus, we know that the sum must at least be $2(n - 1)$. Now, we can show this is possible by providing a single sequence that achieves this sum. Let $a_i = i$, for $1 \le i \le n$. Then our desired sum is $(2 - 1) + (3 - 2) + \cdots \big(n - (n - 1)\big) + |1 - n| = 1 + 1 + 1 + \cdots + 1 + (n - 1) = (n - 1) + (n - 1) = 2(n - 1)$. Intuitively, this sequence "works" because when we move from 1 to n, we always move in the correct direction, always increasing without ever taking a detour back down. Similarly, when we go from n to 1, we go straight back down, without any detours. Thus, we achieve our minimum by never getting side-tracked and never moving in the "incorrect" direction.

b) How many permutations achieve this minimum value?

As noted previously, any permutation can be shifted into any one of n orders, all of which are equivalent in their corresponding sum since they are cyclic shifts of one another. Thus, we'll only consider the $(n - 1)!$ permutations that start with 1 ($a_1 = 1$), and then take this result and multiply it by n, since all n cyclic shifts of any permutaiton that starts with 1 with the desired sum will create a different permutation that satisfies the criterion.

We can place the value n in any slot a_i where $2 \le i \le n$. If we set $a_i = n$, then we are free to ***<u>choose</u>*** $i - 2$ out of the remaining $n - 2$ values to fill in the slots $a_2, a_3, \dots a_{i-1}$. BUT, once we make this choice, the ***<u>order</u>*** of these items is fixed within these slots. For example, for $a_5 = n$, if we choose the subset $\{3, 6, 7\}$ as three items out of the remaining $n - 2$, we MUST have $a_2 =$

3, $a_3 = 6$, and $a_4 = 7$. Any other choice would force a movement of more than $n - 1$ from a_1 to a_5. Once we have these numbers chosen, all of the remaining numbers are ***forced*** to be placed in descending order for $a_{i+1}, a_{i+2}, \ldots a_n$. Thus, our answer to this subproblem, the number of permutations with $a_1 = 1$ that satisfy the criterion is equal to the value of the following summation:

$$\sum_{i=2}^{n} \binom{n-2}{i-2} = \sum_{i=0}^{n-2} \binom{n-2}{i} = 2^{n-2}$$

To get our final result, just multiply this by n. Thus there are $n2^{n-2}$ total permutations that minimize the sum given.

3) A bag contains n pairs of distinct matching tiles. We then start pulling out tiles from the bag at random, one by one. Any time we pull the matching tile to one of the tiles in our hand, we set aside the match. The game stops if we are ever holding three non-matching tiles at the end of a turn, or if we've pulled all of the tiles from the bag. If the latter occurs, we've won the game. What's the probability of winning the game, in terms of n?

<u>Solution</u>
Note that if $n < 3$, the probability of winning is 1. The analysis below assumes $n \geq 3$.

Let p_n be the probability of winning the game when there are n pairs of distinct matching tiles in the bag. We first pull one tile, leaving 2n - 1 tiles in the bag. From this state, consider our next two pulls. The first pull could match our tile and the second would simply be a new tile. This occurs with probability $\frac{1}{2n-1}$, since there is only one match to our original tile out of the 2n - 1 remaining ones. Alternatively, we could pull a non-matching tile followed by a tile that matches one of the first two. This occurs with probability $\frac{2n-2}{2n-1} \times \frac{2}{2n-2} = \frac{2}{2n-1}$, since all but one of the remaining 2n - 1 tiles isn't a match to the first pulled and the second tile pulled must be one of the two held in the hand out of the 2n - 2 remaining at that point. From this point on, we are playing the game with n - 1 matching tiles. Thus, we have the following recurrence relation for p_n:

$$p_n = \left(\frac{1}{2n-1} + \frac{2}{2n-1}\right) p_{n-1} = \frac{3}{2n-1} p_{n-1}$$

Note that $p_2 = 1$ and we can "unroll" the recursive formula by repeatedly plugging into it, getting:

$$p_n = \prod_{k=3}^{n} \frac{3}{2k-1}$$

With some algebra, we can express this quantity as follows: $\frac{3^{n-1}2^n n!}{(2n)!}$.

4) Find the set of integers n that satisfy the following equation: $\sum_{i=45}^{133} \frac{1}{\sin(i°)\sin((i+1)°)} = \frac{1}{\sin(n°)}$

Solution

This question consisted an error. The sum should have gone to 89.

The key to this question is working out the following identity:

$$\cot(i°) - \cot(i+1)° = \frac{\cos(i°)}{\sin(i°)} - \frac{\cos((i+1)°)}{\sin((i+1)°)} = \frac{\sin((i+1)°)\cos(i) - \cos((i+1)°)\sin(i)}{\sin(i)\sin((i+1)°)}$$

$$= \frac{\sin((i+1)-i)}{\sin(i)(\sin(i+1))} = \frac{\sin(1)}{\sin(i)(\sin(i+1))}$$

It then follows that: $\frac{\cot(i°)-\cot((i+1)°)}{\sin(1°)} = \frac{1}{\sin(i°)(\sin((i+1)°))}$.

Here is the corrected problem worked out:

$$\sum_{i=45}^{89} \frac{1}{\sin(i°)\sin((i+1)°)} = \sum_{i=45}^{89} \left(\frac{\cot(i°)-\cot((i+1)°)}{\sin(1°)}\right)$$

$$= \frac{1}{\sin(1°)} \sum_{i=45}^{89} (\cot(i°) - \cot((i+1)°))$$

$$= \frac{1}{\sin(1°)}(\cot(45°) - \cot(90°)) = \frac{1}{\sin(1°)}$$

for this problem, $n \in \{1 + 360k, 179 + 360k \mid k \in Z\}$, since there are two angles, A, in between 0º and 360º, such that sin(A) = sin(1º).

With the original bounds of summation, we can simply work out the sum, which is

$$= \frac{1}{\sin(1°)}(\cot(45°) - \cot(134°)) = \frac{1 - \cot(134°)}{\sin(1°)}$$

There aren't any integers, n for which this expression equals $\frac{1}{\sin(n°)}$. We can see this because the maximal value of this expression without dividing by zero is $\frac{1}{\sin(1°)}$, and the expression above exceeds this value since $\cot(134°)$ is negative.

5) Let the set N_d be the set of divisors of the positive integer N. With proof, determine $\sum_{k \in N_d} \phi(k)$, where ϕ represents the Euler phi function.

Solution

Recall that $\phi(N)$ represents the number of integers in the set {1, 2, 3, ..., N-1} that do not share any common factors with N. Let S(N) be the subset of integers of {1, 2, ..., N-1} that share no common factors with N. Now, consider each of the sets S(N_d). For each set S(N_d), define a corresponding set T(N_d) such that iff $k \in S(N_d)$, then $\frac{Nk}{N_d} \in T(N_d)$. We will show that for each item in {1, 2, 3, ..., n-1} it belongs to precisely one set of the form T(N_d). Since all items from these sets must come from {1, 2, 3, ..., n-1} it follows that $\sum_{k \in N_d} \phi(k) = N - 1$, since there are N-1 items in {1, 2, ..., N-1}.

Before we dig into the full proof, it might be useful to take a look at an example, N = 12. It's divisors are 1, 2, 3, 4, 6 and 12. Here are the corresponding sets S and T for each of these values:

Divisor	S	T
1	∅	∅
2	{1}	{6}
3	{1, 2}	{4, 8}
4	{1, 3}	{3, 9}
6	{1, 5}	{2, 10}
12	{1, 5, 7, 11}	{1, 5, 7, 11}

Note that each value, 1 through 11, appears exactly once in the column labeled T.

Consider any item $k \in \{1,2,3, \ldots, N-1\}$. Let $c = \gcd(k, N)$. It follows that $\gcd\left(\frac{k}{c}, \frac{N}{c}\right) = 1$, by the definition of gcd. This proves that $\frac{k}{c} \in S(\frac{N}{c})$. By, definition of $T(N_d)$, it follows that $k \in T(\frac{N}{c})$. Since $\frac{N}{c} \in N_d$, we've shown that every value of k belongs to at least one of the sets of the form T(N_d).

Now, let's prove that every item $k \in \{1,2,3, \ldots, N-1\}$, can not belong to two distinct sets of the form T(N_d). We've show that that $k \in T(\frac{N}{c})$. Now, we must show that it can't be a member of any other set T. All of the sets T can be expressed as $T(\frac{N}{d})$, where d is a divisor of N with $d \neq c$. For k to belong in this set, we must have that $d \mid k$. It follows that $d \mid \gcd(k, N)$. Namely, $d \mid c$. Combining this with the fact that $d \neq c$, we can conclude that d is a proper divisor of c. If this is the case, then we know that $\frac{N}{d}$ and $\frac{k}{d}$ share a common factor of $\frac{c}{d} > 1$. Thus, by definition of the set S, $\frac{k}{d} \notin S(\frac{N}{d})$, thus, we can conclude that $k \notin T(\frac{N}{d})$, as desired.

To help follow along with the proof, try plugging in $k = 8$ and $N = 12$. In this case, $c = 4$, since gcd(8/4, 12/4) = gcd(2, 3) = 1. Thus, 2 ∈ S(3), implying that 8 ∈ T(3), as the table illustrates. A different value of d that divides evenly into both 8 and 12 is $d = 2$. Since this is a proper divisor of 4, we see that 8/2 = 4 and 12/2 = 6 share a common factor. This means that 4 ∉ S(6), so 8 ∉ T(6).