University of Central Florida
Online Mathematics Contest
Solutions: January 2014 (Year 1, Round 1)

Warm Up Solutions

1) Johnny walks to from home to school at a pace of 3 miles per hour. On the same day, excited to get home, he runs home from school at a pace of 8 miles per hour. What was his average speed in miles per hour for this round trip?

Solution

Let d be the distance between home and school. Let r_1 represent Johnny's rate traveling to school, in miles per hour and r_2 represent his rate traveling back home. Let t_1 represent the amount of time he took to get to school and t_2 represent the amount of time he took to return from school. Since distance is the product of rate and time, we solve for both time variables as follows:

$$d = r_1 t_1, \text{ so } t_1 = \frac{d}{r_1} \text{ and } d = r_2 t_2, \text{ so } t_2 = \frac{d}{r_2}$$

Let r equal the average speed for the round trip. Plugging into the distance equation, we get the following, noting that the round trip has length 2d:

$$2d = r\left(\frac{d}{r_1} + \frac{d}{r_2}\right)$$
$$2 = r\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$
$$2 = r\left(\frac{r_1 + r_2}{r_1 r_2}\right)$$
$$r = \left(\frac{2r_1 r_2}{r_1 + r_2}\right)$$

For our specific problem, by plugging in $r_1 = 3$ and $r_2 = 8$, we get our average speed to be $\frac{48}{11}$ mph. Note: In general, the answer to this question is the Harmonic Mean of the two different speeds given. A basic starter on Harmonic Mean can be found here: http://en.wikipedia.org/wiki/Harmonic_mean.

2) Samantha is traveling from city A to city B to city C and back to city A. If she averages 40 mph on the first leg of the trip, 50 mph on the second leg of the trip, 44 mph on the last leg of the trip, the trip from city A to city B was 100 miles, the trip from city B to city C was 200 miles, and the whole trip average was 45 mph, how long was her trip from city C back to city A.

<u>Solution</u>

The trip from A to B took Samantha $\frac{100\ miles}{40\frac{miles}{hour}} = 2.5\ hours$
The trip from B to C took Samantha $\frac{200\ miles}{50\frac{miles}{hour}} = 4\ hours$
If the average for the whole trip is 45, then
$45\ mph = \frac{100\ miles+200\ miles+(44\ mph*x\ hrs)}{2.5\ hrs+4\ hrs+x\ hrs}$

Solving for x, we find that the trip from C to A took 7.5 hours. Therefore, the trip from C to A was 330 miles long. Note that this breaks the triangle inequality, so the path that Samantha chose to take home was longer than traveling back through the original route through which she came.

3) What is the coefficient of a^4b^4 is the expansion of $(2a – 3b)^8$?

<u>Solution</u>

Using the binomial formula, we get this term to be $\binom{8}{4}(2a)^4(-3b)^4 = 70 \times 16 \times 81 \times a^4b^4$. Thus, the desired coefficient is 70 x 16 x 81 = 90720.

4) If $sin\theta = \frac{2}{3}$ and $0 < \theta < 90°$, what are $\sin(2\theta)$, $\cos(2\theta)$, *and* $\tan(2\theta)$.

<u>Solution</u>

If $sin\theta = \frac{2}{3}$ and $0 < \theta < 90°$, using $sin^2\theta + cos^2\theta = 1$, we can find that $cos\theta = \frac{\sqrt{5}}{3}$. (Note: The angle restriction allows us to conclude that $cos\theta$ is positive.)

With these building blocks, we can solve for the desired quantities:

$$\sin(2\theta) = 2sin\theta cos\theta = 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{4\sqrt{5}}{9}$$
$$\cos(2\theta) = 2cos^2\theta - 1 = 2(\frac{\sqrt{5}}{3})^2 - 1 = \frac{10}{9} - 1 = \frac{1}{9}$$
$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = 4\sqrt{5}$$

5) What is $(1 + i)^5(2 – 2i)^5$?

<u>Solution</u>

Demoivre's Theorem states that $(r(cos\theta + isin\theta))^n = r^n(\cos(n\theta) + isin(n\theta))$, where r is a real value and θ is an angle (in radians). We also find that $(cos\alpha + isin\alpha)(cos\beta + isin\beta) = \left(\cos(\alpha + \beta) + isin(\alpha + \beta)\right)$. This question can be most easily solved by converting both complex numbers in it to this form, appropriate for De Moivre's Theorem, applying the theorem and then converting the result back to regular complex form. Note: Since the expressions of the form $cos\theta + isin\theta$ appear frequently in this solution and others, we will use the shorthand $cis\theta$ to refer to it.

In general, this idea is a powerful one in mathematics. The general idea is to transform the representation of a problem from one domain to another, solve the problem in that alternate domain because it's easier to do so than the original domain, and then convert the solution back over to the original domain. An example of a frequently used mathematical tool that falls under this category is MJ this might be beyond the knowledge of most high school students, it's good for future engineers to have some recognition of the term: http://en.wikipedia.org/wiki/Fast_Fourier_transform.

Now, we solve the problem as follows:

$$\begin{aligned}
(1+i)^5(2-2i)^5 &= (\sqrt{2}\left(cis\frac{\pi}{4}\right))^5(2\sqrt{2}(cis(\frac{-\pi}{4}))^5 \\
&= \sqrt{2}^5 cis\frac{5\pi}{4}(2\sqrt{2})^5 cis\frac{-5\pi}{4} \\
&= 2^5\sqrt{2}^{10} cis(\frac{5\pi}{4} + \frac{-5\pi}{4}) \\
&= 2^5 2^5 cis(0) \\
&= 1024
\end{aligned}$$

Exercise Solutions

1) You are given that x + y = 17 and xy = 60. What is $x^3 + y^3$?

Solution

Use the binomial theorem as follows:

$$17^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$x^3 + y^3 = 17^3 - 3x^2y - 3xy^2$$
$$x^3 + y^3 = 17^3 - 3xy(x+y)$$

Using the rest of the given information, we solve as follows:

$$x^3 + y^3 = 17^3 - 3(60)(17) = 1853$$

Incidentally, we can also solve the question be determining that the pair x and y take the values 5 and 12 and cubing both of these. Given a different set of initial values than 17 and 60, this approach could involve more work, especially if the solutions are irrational or complex.

2) How many permutations of "CENTRALFLORIDA" do not contain any double letters? Note: A double letter in a string is the same letter next to itself.

Solution

It is easier to first find out how many total permutations there are of "CENTRALFLORIDA" and then subtract the amount of permutations if there are double letters. Since there are three sets of items to subtract out, we must use the Inclusion-Exclusion Principle.

Here is the overview on Wikipedia:

http://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle

and the Art of Problem Solving Website:

http://www.artofproblemsolving.com/Wiki/index.php/Principle_of_Inclusion-Exclusion

Total amount of permutations of "CENTRALFLORIDA": $\frac{14!}{2!*2!*2!} = 10{,}897{,}286{,}400\ ways$

Now, we must count the number of permutations with at least one set of double letters. We have three sets of double letters: 1)"AA", 2)"LL", 3)"RR"

Thus, in counting the permutations that have at least one of these, if we let these sets be A, B and C, respectively, the value of the total number of permutations, using the Inclusion-Exclusion Principle is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Luckily, for this question the sizes of the first three sets are the same, as well as the second and third sets. The following argument works for all three sets, A, B and C. In order to calculate the number of permutations with the double letter "AA", for example, treat "AA" as a single letter. Then we have 13 letters instead of 14 and 2 repeated letters. Using the permutations with repetition formula, we get:

$$|A| = \frac{13!}{2! * 2!} = 1{,}556{,}755{,}200 = |\mathrm{B}| = |\mathrm{C}|$$

Now, let's count sets that contain both "AA" and "LL". Here, we just create two "superletters", "AA" and "LL", so that we are really permuting only 12 letters with 1 repeated letter, "R". It follows that:

Fourth, fifth, and sixth case: $|A \cap B| = \frac{12!}{2!} = 239{,}500{,}800 = |\mathrm{A} \cap \mathrm{C}| = |\mathrm{B} \cap \mathrm{C}|$

Finally, we need to count the number of permutations with all three pairs of double letters. Treating each as a "superletter", we have 11 distinct objects to permute, so there are 11! Such permutations.

Plugging into our whole formula, we get the total number of permutations with at least one repeated double letter to be:

$$3 * \frac{13!}{2! * 2!} - 3 * \frac{12!}{2!} + 11!$$

We can finally conclude the answer to the question by subtracting this value from the total number of permutations previously calculated:

$$\frac{14!}{2! * 2! * 2!} - \left(3 * \frac{13!}{2! * 2!} - 3 * \frac{12!}{2!} + 11!\right) = 6{,}905{,}606{,}400$$

3) What are the sums of the roots of the equation $4(81^x) - 43(9^x) + 108 = 0$?

<u>Solution</u>
Let $y = 9^x$ and substitute accordingly:

$$4y^2 - 43y + 108 = 0$$
$$(4y - 27)(y - 4) = 0$$
$$So, y = \frac{27}{4} or\ y = 4. Thus, 9^x = \frac{27}{4}\ or\ 9^x = 4.$$

The sum of the values of x that satisfy the two given equations is

$$log_9 \frac{27}{4} + log_9 4 = log_9 \left(\frac{27}{4} \times 4\right) = log_9 27 = \frac{3}{2}$$

In many questions, we can avoid solving for the different values of y and just utilize the fact that the sums of the roots of a quadratic equation $ax^2 + bx + c = 0$ is $-\frac{b}{a}$, but for the purposes of this question, since we didn't want the sum of the values of y, we had to go ahead and solve for the two different solutions for y. Note that we used log rules to avoid solving for the individual solutions for x.

4) In a group project, there is a "to-do" list with 13 different items. Seven of the items involve writing reports while the other six involve building a robot. In a group of four students, if everyone must write at least one report, how many ways can you distribute the work? We count two distributions of work differently if at least one student in the two distributions has a different *number* of a particular task. *Don't* distinguish between distributions where each student has the same number of reports to write and robot tasks in both distributions. (As an example, one possible distribution would be student A writes 2 reports and does 2 robot tasks, student B writes 2 report and does 2 robot tasks, student C writes 2 reports and does 2 robot tasks, and student D writes 1 report and does no robot tasks. This particular distribution should only be counted once, no matter *which* reports student A writes. In short, the students are distinguishable, but the reports are indistinguishable and the robot tasks are indistinguishable from one another.)

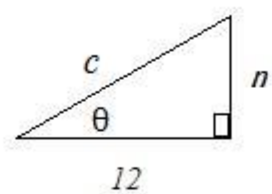
<u>Solution</u>
Out of the 7 reports, we must give 1 to each of the four students, so there are only 3 reports left to distribute. We are counting combinations with repetition for this portion of the problem. In particular, we have 3 items left to give to 4 people. In general, if we are distributing n indistinguishable objects amongst k people, we can do so in $\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}$ ways. Thus, we can distribute the last three reports in $\frac{(3+4-1)!}{3!\times(4-1)!} = 20\ ways$.

Similarly, we can distribute 6 robot tasks to 4 people in $\frac{(4+6-1)!}{6!\times(4-1)!} = 84\ ways$. Since the choice of the first distribution is independent of the second, we multiply to find the total number of ways to distribute the work. Thus, we get 20 x 84 = 1,680.

5) Let θ be any acute angle such that $tan\theta = \frac{n}{12}$, for some positive integer n. For how many different values of θ will sin θ be rational?

Solution

If θ is acute, then we can use the right triangle definition of tangent to draw a picture with theta, with the sides of a right triangle appropriately labeled:



In this triangle, we find that $sin\theta = \frac{n}{c}$. Since n is rational and we want sin θ, to be rational, it follows that we want c to be rational as well, since $c = \frac{n}{sin\theta}$, and dividing two rational numbers yields another.

Using the Pythagorean Theorem, we get:

$$12^2 + \mathrm{n}^2 = c^2$$
$$c^2 - \mathrm{n}^2 = 144$$
$$(\mathrm{c}-\mathrm{n})(\mathrm{c}+\mathrm{n}) = 2^4 3^2$$

It can be shown that the square of any non-integer rational number is also not an integer. (The proof of this will be shown at the end of this problem to preserve continuity.) Since n is a positive integer, we can conclude that the left hand side of the initial equation is also an integer. Thus, c^2 is integral. But, if c were NOT integral itself, c^2 would not be integral. It follows that the only possibility is that c is an integer as well. Thus, c – n and c + n are integers. Since there are 15 divisors of 144, we can write down 7 products of two distinct integers that multiply to 144. Noting that c – n < c + n because both c and n are positive, we set up the following equations:

(1)	c – n = 1, c + n = 144	→	2c = 145
(2)	c – n = 2, c + n = 72	→	2c = 74
(3)	c – n = 3, c + n = 48	→	2c = 51
(4)	c – n = 4, c + n = 36	→	2c = 40
(5)	c – n = 6, c + n = 24	→	2c = 30
(6)	c – n = 8, c + n = 18	→	2c = 26
(7)	c – n = 9, c + n = 16	→	2c = 25

For each of these equations, we can add the two equations to yield a sum for 2c. Equations (2), (4), (5), and (6) yield the following four integral solutions for c: 37, 20, 15 and 13, respectively. The corresponding values for n are 35, 16, 9, and 5. Note that each of these corresponds to Pythagorean Triples. Incidentally, there is a formula to generate all primitive Pythagorean Triples and that formula could also have been used to determine the number of angles that satisfy the given requirement. Read more here: http://en.wikipedia.org/wiki/Pythagorean_triple.

Investigations

1) You are given 3^n coins, where n is a positive integer. All of these coins except 1, which is heavier than the rest, weigh the exact same. Using mathematical induction, prove that n weighings on a balance are sufficient to determine which coin is the heavy one. Once again, using mathematical induction, show that n – 1 weighings are insufficient to always determine the heavy coin.

Solution

First, let's prove that n weighings are sufficient, using induction on n.

Base case: n = 0. Given $3^0 = 1$ coin, exactly one of which is heavy, 0 weighings can determine the heavy coin, since the only coin you have must be heavy.

Inductive hypothesis: For an arbitrary positive integer n = k, assume that k weighings are sufficient to determine the heavy coin out of 3^k coins.

Inductive step: Prove, for n = k+1, that k+1 weighings are sufficient to determine the heavy coin out of 3^{k+1} coins.

To prove the inductive step, consider being given 3^{k+1} coins. Split these arbitrarily into three equal sized groups, A, B and C. Each group must have $3^{k+1}/3 = 3^k$ coins in it. Do one weighing, putting all of the coins in group A of one side of the scale and all of the coins in group B on the other side of the scale. If these balance, the heavy coin must be in group C. If they don't balance, the heavier side of the two must contain the heavy coin. No matter which outcome occurs, in one weighing, we have determined which of the three groups contains the heavy coin. By our inductive hypothesis, we can use k successive weighings on this group to uniquely determine the heavy coin. Thus, in k+1 weighings, we can determine the heavy coin, proving the inductive step.

Next, let's prove that n-1 weighings are insufficient, using induction on n.

Base case: n = 1. Given 3^1 coins, 1 – 1 = 0 weighings is clearly insufficient to determine the heavy coin, since if we have more than 1 coin, at least one weighing must occur to differentiate which one it is.

Inductive hypothesis: For an arbitrary positive integer n = k, assume that k – 1 weighings are insufficient to determine the heavy coin out of 3^k (or more) coins.

Inductive step: Prove, for n = k+1, that k weighings are sufficient to determine the heavy coin out of 3^{k+1} (or more) coins.

To prove the inductive step, let's consider all actions we could do with 3^{k+1} coins. We know that if we try to balance two sets of coins with different numbers of coins in each set, the balance information tells us nothing, since we don't know how much heavier the heavy coin is. Thus, any reasonable weighing must have an equal number of coins on both sides of the balance. Let this be x coins. Thus, we have split our coins into three groups, of sizes x, x and $3^{k+1} - 2x$. It's fairly easy to see that if $x < 3^k$, then $3^{k+1} - 2x > 3^k$, if $x > 3^k$, $3^{k+1} - 2x < 3^k$, and if $x = 3^k$, $3^{k+1} - 2x = 3^k$. In all cases, at least one of the three groups contains 3^k or more coins.

Since we don't know in which of the three groups the heavy coin is in advance, based on our first weighing, the heavy coin could theoretically end up in any of the three groups. Since our procedure must always identify the heavy coin, no matter what the result of the weighings is, there is a possibility that the heavy coin will be in one of the groups of size 3^k or more. If this happens, our inductive hypothesis can be invoked to prove that $k - 1$ weighings will not determine which coin is the heavy coin, in all cases. Adding, $1 + (k - 1) = k$, we have shown that it will not be possible in all cases to determine the heavy coin out of 3^{k+1} original coins in k weighings, proving the inductive step.

2) Elise and Ellie are identical twins who really like powers of two. Not so surprisingly, they share a car with a six-digit odometer and like odometer readings that only contain digits that are perfect powers of 2. (Thus, the first odometer reading they like is 111111 and the 8^{th} one is 111128, for example.) Determine the 2013^{th} odometer reading they like.

Solution

Four digits are powers of 2 (1, 2, 4 and 8), thus, for each slot on the odometer, there are 4 possible choices. In short, we can make a one-to-one correspondence between numbers in base 4 and odometer readings. For example, the number 000123_4 corresponds to the odometer reading 111248. Since the first odometer reading the twins like corresponds to the base 4 value of 0, we want the odometer reading that corresponds to the base 4 value of 2012. We must subtract one from the rank, since the rank numbers are 1-based and the base 4 numbers are 0-based.

Converting 2012_{10} to base 4 we obtain:

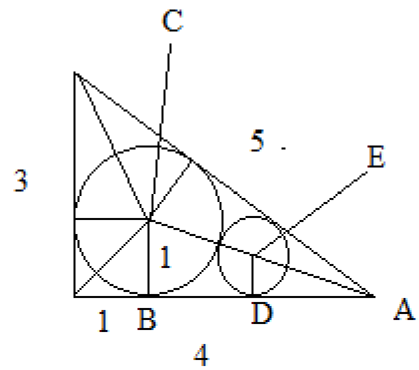
2012 divided by 4 = 503, remainder 0
503 divided by 4 = 125, remainder 3
125 divided by 4 = 31, remainder 1
31 divided by 4 = 7, remainder 3
7 divided by 4 = 1 remainder 3
1 divided by 4 = 0 remainder 1

Reading the remainders from bottom to top we find that $2012_{10} = 133130_4$. This corresponds to the odometer reading, 288281, the 2013^{th} odometer reading that Elise and Ellie like.

3) Let circle C be the inscribed circle in a right triangle with side lengths 3, 4 and 5. Now, create circles X, Y and Z where each circle is tangent to two sides of the triangle and C. What is the sum of the radii of X, Y and Z?

Solution

We can use similar triangles to help us find each of the radii. Consider the whole figure with one of the circles highlighted:



Note that we can obtain the inradius, r, of this triangle by seeing that the area of the whole triangle can be viewed as the sum of the three triangles that have as their height the inradius and their base each of the three sides of the triangle. In this manner, we see we can that the area of this triangle can be expressed as $\frac{1}{2}(3r)+\frac{1}{2}(4r)+\frac{1}{2}(5r)=6r$. Since the area of this right triangle is 6, we see that the inradius is 1. More generally, for any triangle, we find that its inradius is $\frac{A}{s}$, where *A* is the area of the triangle and *s* is its semiperimeter.

In this specific picture, triangles EDA ane CBA are similar, with |AB| = 3 and |BC| = 1, so |AC| = $\sqrt{10}$. Without loss of generality, let r_X = |ED|. Set up a ratio between the shortest and longest sides of these to similar triangles, noting that we must have |EA| = $\sqrt{10}r_X$:

$$\frac{r_X}{1}=\frac{|EA|}{|CA|}=\frac{|EA|}{|EA|+1+r_X}=\frac{\sqrt{10}r_X}{\sqrt{10}r_X+1+r_X}$$

$$\sqrt{10}r_X+1+r_X=\sqrt{10}$$
$$(\sqrt{10}+1)r_X=\sqrt{10}-1$$
$$r_X=\frac{\sqrt{10}-1}{\sqrt{10}+1}=\frac{\sqrt{10}-1}{\sqrt{10}+1}\times\frac{\sqrt{10}-1}{\sqrt{10}-1}=\frac{11-2\sqrt{10}}{9}$$

Note that when we look at the other two corners, our pictures are similar, except we replace the side |AB| = 3 with |AB| = 2 and |AB| = 1, leading to |AC| = $\sqrt{5}$ and |AC| = $\sqrt{2}$, respectively. Thus, we find the sum of the radii is:

$$r_X+r_Y+r_Z=\frac{\sqrt{10}-1}{\sqrt{10}+1}+\frac{\sqrt{5}-1}{\sqrt{5}+1}+\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{11-2\sqrt{10}}{9}+\frac{3-\sqrt{5}}{2}+3-2\sqrt{2}\sim 1.07303274$$

4) Let $f(x) = x^3 – ax^2 + bx – c = 0$ with non-zero real roots, of which exactly 2 are equal to r_1, with $a^2 – 4b = 0$. What are a, b and c in terms of r_1?

Solution

Let r_2 be the non-repeated root of f(x). Using the roots of f(x), we have

$$f(x) = (x – r_1)(x – r_1)(x – r_2)$$

Equating the two expressions of f(x), we have
$$x^3 – ax^2 + bx – c = x^3 – (2r_1 + r_2)x^2 + (r_1^2 + 2r_1r_2)x – r_1^2r_2$$

Equating coefficients in this equation, we have

$$a = 2r_1 + r_2 \qquad b = r_1^2 + 2r_1r_2 \qquad c = r_1^2r_2$$

Using the given information, we have:

$$a^2 – 4b = (2r_1 + r_2)^2 – 4(r_1^2 + 2r_1r_2) = 4r_1^2 + 4r_1r_2 + r_2^2 – 4r_1^2 – 8r_1r_2 = r_2^2 – 4r_1r_2 = r_2(r_2 – 4r_1) = 0$$

Noting that $r_2 \neq 0$ and that $r_2(r_2 – 4r_1) = 0$, it follows that $r_2 – 4r_1 = 0$ and $r_2 = 4r_1$.

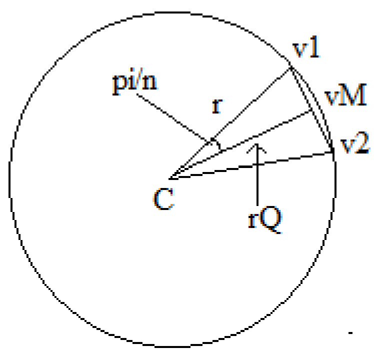
Now, plugging back for r_2 in the equations for a, b and c yields:

$a = 2r_1 + r_2 = 2r_1 +4r_1 = 6r_1$.
$b= r_1^2 + 2r_1r_2 = r_1^2 + 2r_1(4r_1) = 9r_1^2$.
$c = r_1^2r_2 = r_1^2(4r_1) = 4r_1^3$.

5) Let P_n be a regular polygon with unit area of n sides for any integer $n \geq 3$. Define Q_n to be the regular polygon created by connecting the midpoints of each side of P_n. In terms of n, what is the area of Q_n?

Solution

The critical part of the figure to consider is a sector that is one-n^th of the whole circle:



Let C be the center of the polygon P_n and let v_1 and v_2 be two consecutive points on P_n. Let r be the length of the radius of the circumscribed circle of P_n. The length of the segment connecting C to v_1 is r. P_n can be divided into n triangles congruent to triangle Cv_1v_2. Let v_M be the midpoint of v_1v_2. The area of each of these triangles can be expressed as $\frac{1}{2}r^2 \sin\left(\frac{2\pi}{n}\right)$. Since we know that n of these triangles have unit area together, we get the following relationship:

$$A(P_n) = 1 = \frac{n}{2}r^2 \sin\left(\frac{2\pi}{n}\right).$$

The apothem of P_n represents a connection between C and v_M, one of the vertices of Q_n. If r is the inradius of P_n, then we can find an expression for r_Q, the inradius of Q_n, using the right triangle Cv_1v_M. The angle $v_1Cv_M = \frac{\pi}{n}$ radians. Thus, we get the equation:

$$\cos\left(\frac{\pi}{n}\right) = \frac{r_Q}{r}$$

Solving for r_Q, we find $r_Q = rcos(\frac{\pi}{n})$. We can find the area of Q_n:

$$A(Q_n) = \frac{n}{2}r_Q^2 \sin\left(\frac{2\pi}{n}\right) = \frac{n}{2}(rcos(\frac{\pi}{n}))^2 \sin\left(\frac{2\pi}{n}\right) = A(P_n)cos^2\left(\frac{\pi}{n}\right) = cos^2\left(\frac{\pi}{n}\right)$$

Note: If we make the observation that if the ratio between any corresponding one-dimensional items in two similar figures is x, then the ratio between their areas is x^2, we can arrive at our answer by only analyzing the triangle Cv_1v_M as we did at the end or solution.