University of Central Florida
Online Mathematics Contest
Problems: January 2015 (Year 2, Round 2)

Warm Ups

1) If x is 25% larger than z, and y is 50% larger than z, then y is what percent larger than x?

Solution
Using the given information, we get the following two equations:

$$x = 1.25z \text{ and } y = 1.5z$$

We can solve for z in the first equation, yielding $z = \frac{x}{1.25}$, and substitute this into the second equation: $y = 1.5z = 1.5\left(\frac{x}{1.25}\right) = 1.2x$. It follows that y is 20% larger than x.

2) Given that $x^2 + y^2 = 12x + 4y - 40$, what is x + 2y?

Solution

$$x^2 + y^2 = 12x + 4y - 40$$
$$x^2 - 12x + 36 + y^2 - 4y + 4 = 0$$
$$(x-6)^2 + (y-2)^2 = 0$$

Given that x and y are real numbers, we must have x = 6 and y = 2. It follows that x + 2y = 6 + 2(2) = 10.

3) Three standard six-sided dice are rolled. What is the probability that all three values showing are unique?

Solution
Imagine rolling the dice in sequence. The probability the first die roll is different than all previous ones is 1. The probability the second die roll is different than the previous one is $\frac{5}{6}$. Finally, given that the first two rolls were unique, the probability the last die rolled is different than the first two is $\frac{4}{6}$. Multiplying these conditional probabilities together yields the probability that all three dice show unique values. Thus, the desired probability is $1 \times \frac{5}{6} \times \frac{4}{6} = \frac{5}{9}$.

4) Two cars start 100 miles apart, driving towards each other. The first car drives at a steady rate of 30 mph while the second car drives at a steady rate of 20 mph. A bird, flying at 55 mph starts at the first car and flies until it reaches the second car, then reverses direction and goes back to the first car, and so forth, until the cars meet. Assuming that the bird can instantaneously change direction without changing speed, how far did the bird fly in the time that it took the two cars to meet?

Solution

In two hours the first car covers 60 miles and the second car covers 40 miles. Thus, in two hours time, the two cars meet. In these two hours, the bird has flown 55 mph x 2 hours = 110 miles.

5) Jenny has some Pokemon cards she wants to distribute to her friends. If she tries to give an equal number of cards to each of five friends, she's left with two cards. If she tries to give an equal number of cards to each of seven friends, she's left with three cards. If she tries to give an equal number of cards to each of nine friends, she's left with four cards. If Jenny has less than 1000 cards, list all possible number of cards Jenny could have.

Solution

Using the given information, if we let x be the number of cards Jenny has, we get the following equations:

$$x \equiv 2 (mod 5)$$
$$x \equiv 3 (mod 7)$$
$$x \equiv 4 (mod 9)$$

A formal solution of this problem requires the Chinese Remainder Theorem (http://en.wikipedia.org/wiki/Chinese_remainder_theorem).

Since this is unlikely to be known by many, we'll derive an alternate solution here.

For integers a, b and c, the given equations can be rewritten as

$$x = 5a + 2$$
$$x = 7b + 3$$
$$x = 9c + 4$$

We start with the first two equations:

$$7b + 3 = 5a + 2$$
$$5a = 7b + 1$$

The smallest non-negative integer b that satisfies this equation is b = 2, for which the corresponding value of a is 3. The next value of b that satisfies this equation is b = 7, while the corresponding solution for a = 10. In general, we can see that we can generate more solutions from the original base solution by adding 5 to a previous solution for b and 7 to the corresponding solution for a. Logically, this makes sense since both additions add 35 to both sides of the equation, keeping it balanced. A small brute force shows that for these specific numbers, no smaller offset can be added to both sides of the equation to keep it balanced.

These solutions indicate that $x \equiv 17 (mod\ 35)$. Now, for some integers c and d, we have:

$$x = 35d + 17$$
$$x = 9c + 4$$

Setting these equal to one another we get:

$$9c + 4 = 35d + 17$$
$$9c = 35d + 13$$

Substituting integers for d, starting with 0, we get the value of the right hand side to be 13, 48, 83, 118, and 153. This last value is divisible by 9, so we get a solution of d = 4, c = 17.

Using the same logic as before, we find that $x \equiv 157 (mod 315)$.

To finish the question, we must simply list each value of x that satisfies this equivalence that is less than 1000. These are 157, 472, and 787.

Exercises

1) Let a and b be positive integers with a ≤ b such that their greatest common divisor is 12 and their least common multiple is 4320. What are all of the possible values of the ordered pair (a, b)?

Solution
Using the Fundamental Theorem of Arithmetic, we can prove that the product of the gcd and lcm of two numbers is equal to the product of the two numbers themselves. Thus, we have

$$ab = 12 \times 4320 = 2^7 3^4 5^1.$$

We also know that both a and b are divisible by 12. Thus, let a = 12x and b = 12y, for integers x amd y. It follows that gcd(x, y) = 1 and xy = $2^3 3^2 5^1$. In order to maintain gcd(x, y) = 1 restriction, we can assign x and y as follows:

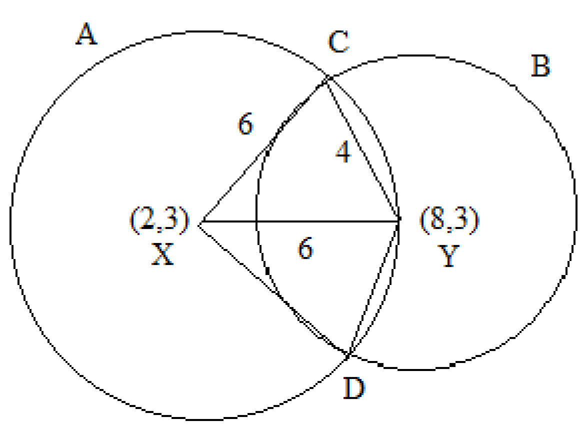
$$\begin{gathered} x = 1, y = 2^3 3^2 5 \\ x = 2^3, y = 3^2 5 \\ x = 3^2, y = 2^3 5 \\ x = 5, y = 2^3 3^2 \end{gathered}$$

Essentially, the gcd restriction forces us to assign x to a subset of the unique prime factors 2, 3 and 5. Since a ≤ b, we must restrict our choice of subsets to be less than or equal to the square root of $2^3 3^2 5^1$. It follows that we have four solutions that satisfy the given information:

(12, 4320), (60, 864), (96, 540), and (108, 480).

2) Circle A is centered at (2, 3) with a radius of 6 and circle B is centered at (8, 3) with a radius of 4. What is the area of the intersection of circles A and B?

Solution
Given the radii and that the distance between the centers of the circles is 6, we get the following picture:



Use the Law of Cosines on triangle CXY triangle to find half of the central angles of both sectors, angles CXY and CYX. Let α be angle CXY and β be angle CYX. We find:

$$cos\alpha = \frac{6^2 + 6^2 - 4^2}{2(6)(6)} = \frac{7}{9}$$
$$cos\beta = \frac{6^2 + 4^2 - 6^2}{2(6)(4)} = \frac{1}{3}$$

To find the area of the intersection, we must split the work into two pieces: Finding the area to the right of the line segment CD and the are to the left of the line segment CD. We find the former by taking the area of sector CXD and subtracing the area of triangle CXD from that. We find the latter by taking the area of sector CYD and subtracting the area of triangle CYD. Both of these regions are called circular segments.

First, we must find the sine and cosine of *2α* and *2β*:

$$\sin(2\alpha) = 2sin\alpha cos\alpha = 2\sqrt{1 - (\frac{7}{9})^2}\left(\frac{7}{9}\right) = \frac{56\sqrt{2}}{81}$$
$$\cos(2\alpha) = 2cos^2\alpha - 1 = 2\left(\frac{7}{9}\right)^2 - 1 = \frac{17}{81}$$

$$\sin(2\beta) = 2sin\beta cos\beta = 2\sqrt{1 - (\frac{1}{3})^2}\left(\frac{1}{3}\right) = \frac{4\sqrt{2}}{9}$$
$$\cos(2\beta) = 2cos^2\beta - 1 = 2\left(\frac{1}{3}\right)^2 - 1 = -\frac{7}{9}$$

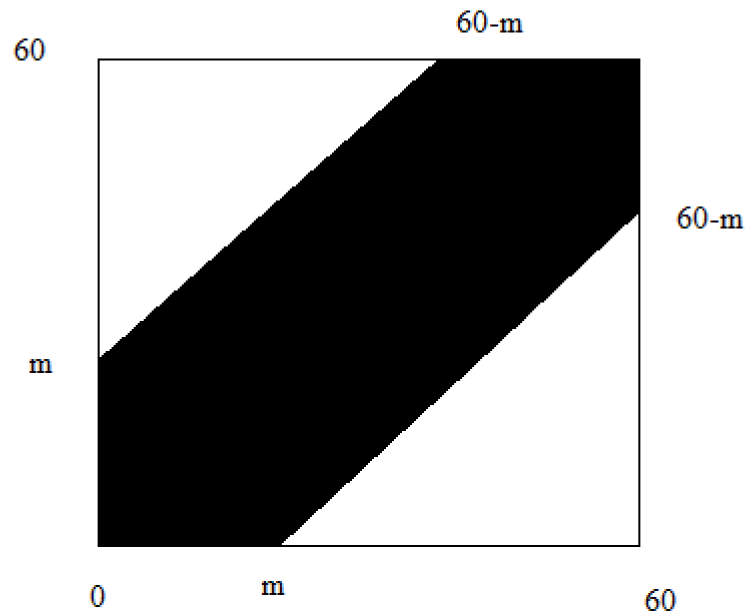
In general, the area of a circular segment in a circle with radius r and central angle θ is $\frac{\theta}{2}r^2 - \frac{1}{2}r^2 sin\theta = \frac{1}{2}r^2(\theta - sin\theta)$. Applying this formula to our specific problem to the two circular segment areas we must sum, we get the following:

$$Intersection = \frac{1}{2}6^2\left(cos^{-1}\left(\frac{17}{81}\right) - \frac{56\sqrt{2}}{81}\right) + \frac{1}{2}4^2(cos^{-1}\left(-\frac{7}{9}\right) - \frac{4\sqrt{2}}{9})$$
$$= 18\left(cos^{-1}\left(\frac{17}{81}\right) - \frac{56\sqrt{2}}{81}\right) + 8(cos^{-1}\left(-\frac{7}{9}\right) - \frac{4\sqrt{2}}{9})$$
$$\sim 21.54$$

3) Two professors arrive at the break room, randomly, in between 10 am and 11 am. If each stays in the break room for exactly *m* minutes, the probability that they run into each other in the break room is .6. What is the value of *m*?

Solution

Let's visualize the sample space of when the two professors arrive as a square in the Cartesian plane. Label both the x and y axes of the square from 0 to 60 minutes and let x be a randomly chosen value in between 0 and 60 representing when the first professor arrives and let y be a randomly chosen value in between 0 and 60 when the second professor arrives. It follows that every point within the square is a point in the sample space. Based on the problem specification, we can determine which areas of the square correspond to the professors meeting and which ones don't. The times that are within *m* minutes of each other are represented in the middle "strip":



We must set the area of this strip to be .6 of the whole area, 3600 "square minutes". It's easier to find the area of this strip by subtracting the area of the two congruent triangles at the edges from the area of the whole square. We get the following when doing so:

$$3600 - 2\left(\frac{(60-m)(60-m)}{2}\right) = 3600(.6)$$
$$3600 - (60-m)^2 = 2160$$
$$3600 - (3600 - 120m + m^2) = 2160$$
$$120m - m^2 = 2160$$
$$m^2 - 120m + 2160 = 0$$
$$m = \frac{120 \pm \sqrt{120^2 - 4(2160)}}{2}$$

$$m = \frac{120 \pm \sqrt{24^2(5^2 - 15)}}{2}$$

$$m = \frac{120 \pm 24\sqrt{10}}{2}$$

$$m = 60 - 12\sqrt{10}$$

We take the smaller root since the value of *m* must be in between 0 and 60. Thus, both professors take a break of $60 - 12\sqrt{10}$ minutes, which is roughly 22.05 minutes.

4) David is taking a matching test where he is matching n words with n definitions. (Each word maps to exactly one of the definitions given.) Unfortunately, David forgot to study and will generate a random matching of words to definitions. What is the expected number of correct pairings he'll choose?

Solution

There are n! possible answers David could give on his test. Since he's matching at random, we will assume that he's equally likely to give each of these answers. Of these n! answers, (n-1)! of them have the answer to the first question correct. To see this, note that there are exactly (n-1)! permutations that have the first answer fixed to be the correct one, since the other n-1 answers may be permuted in any way. This logic holds for all of the rest of the questions as well. Thus, the probability of getting any of the questions correct is $\frac{(n-1)!}{n!} = \frac{1}{n}$. Using the linearity of expectation, we add up the chance of getting each question correct to get the expected number of correct responses to be $n\left(\frac{1}{n}\right) = 1$.

Another way to view the question might be to consider all n! permutations of answers and add up the total score of all those permutations. (n-1)! of them will have the first question correct, for a total score of (n-1)!, (n-1)! of them will have the second question correct, adding (n-1)! to the total score, and so forth. Thus, to add up the scores of all of these answers, we'd get (n-1)! added n times, which equals n!. To get the average score, we must divide by the number of permutations, which is n!, to get an average score of 1.

5) What is the sum of the real roots of the following equation?

$$3^{49x+3} + 3^{147x} = 3^{98x+3} + 1$$

Solution

Noticing that 98 and 147 are multiples of 49, we use a parameter to simplify the original equation, letting $y = 3^{49x}$. Substituting, we get:

$$3^3 y + y^3 = 3^3 y^2 + 1$$
$$y^3 - 27y^2 + 27y - 1 = 0$$

Let the roots of the original equation be x_1, x_2, and x_3. Based on this equation, the product of 3^{49x_1}, 3^{49x_2}, and 3^{49x_3}, is 1. Setting up this product we find:

$$3^{49x_1}3^{49x_2}3^{49x_3} = 1$$
$$3^{49(x_1+x_2+x_3)} = 3^0$$
$$49(x_1 + x_2 + x_3) = 0$$
$$x_1 + x_2 + x_3 = 0$$

Finally, we must verify that all of the roots of the original equation are real. Using DesCartes law of signs, we find that all three roots of the equation in y are real. Since the range of the function $f(x) = 3^x$ is the set of all reals, it follows that all three roots of this equation are real and our desired sum is 0.

Note: the equation in y can be solved for completely, noting that 1 is a root using synthetic division. From there, we can use the quadratic equation to get the other two roots:

$$y^3 - 27y^2 + 27y - 1 = 0$$
$$(y - 1)(y^2 - 26y + 1) = 0$$

$$y = 1 \text{ or } y = \frac{26 \pm \sqrt{26^2 - 4}}{2} = 13 \pm 2\sqrt{42}$$

The corresponding roots for x are $x = 0$ or $x = \frac{log_3(13+2\sqrt{42})}{49}$ or $x = \frac{log_3(13-2\sqrt{42})}{49}$.

Investigations

1) Suppose that α, β, and γ are angles such that

$$\tan(\alpha+\beta)=2, \tan(\beta+\gamma)=3, \text{ and } \tan(\alpha+\gamma)=4.$$

What are possible value(s) of $\tan(\alpha+\beta+\gamma)$?

Solution

Let $X=\alpha+\beta$, $Y=\beta+\gamma$, and $Z=\alpha+\gamma$. We will first find tan($X + Y + Z$).

$$\tan(X+Y)=\frac{tanX+tanY}{1-tanXtanY}=\frac{2+3}{1-2(3)}=-1$$

$$\tan\big((X+Y)+Z\big)=\frac{\tan(X+Y)+tanZ}{1-\tan(X+Y)tanZ}=\frac{-1+4}{1-(-1)4}=-\frac{3}{5}$$

Note that $X+Y+Z=2(\alpha+\beta+\gamma)$. Thus, we have

$$\tan\big(2(\alpha+\beta+\gamma)\big)=\frac{2\tan(\alpha+\beta+\gamma)}{1-tan^2(\alpha+\beta+\gamma)}=-\frac{3}{5}$$

Let $\mathrm{W}=\tan(\alpha+\beta+\gamma)$. Substituting, we have:

$$\frac{2W}{1-W^2}=-\frac{3}{5}$$

$$10W=3-3W^2$$
$$3W^2+10W-3=0$$
$$W=\frac{-10\pm\sqrt{10^2-4(3)(-3)}}{2(3)}$$
$$W=\frac{-10\pm\sqrt{136}}{2(3)}$$
$$W=\frac{-5\pm\sqrt{34}}{3}$$

Thus, the possible values of $\tan(\alpha+\beta+\gamma)$ are $\frac{-5+\sqrt{34}}{3}$ and $\frac{-5-\sqrt{34}}{3}$.

2) Suppose that α, β, γ, and δ are complex numbers satisfying

$$\alpha+\beta+\gamma+\delta=2,$$
$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=3,$$
$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=5,$$
$$\alpha\beta\gamma\delta=7.$$

Find the values of:
(a) $(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta)$
(b) $(\alpha^2+\beta^2+\gamma^2)(\alpha^2+\beta^2+\delta^2)(\alpha^2+\gamma^2+\delta^2)(\beta^2+\gamma^2+\delta^2)$

Solution

(a) Let f(x) be the polynomial with roots α, β, γ and δ. Using the given equations, we know that

$$f(x)=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=x^4-2x^3+3x^2-5x+7$$

Now, note that $f(\alpha+\beta+\gamma+\delta)=(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta)$.

It follows that the quantity we desire is

$$f(\alpha+\beta+\gamma+\delta)=f(2)=16-16+12-10+7=9$$

Another alternate solution that roughly utilizes the same idea as above, but less elegantly is provided below.

Let $S=\alpha+\beta+\gamma+\delta$. The quantity we desire can be rewritten as $(S-\alpha)(S-\beta)(S-\gamma)(S-\delta)$. Since we know *S = 2*, we can further simplify:

$$(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta)=$$

$$(S-\alpha)(S-\beta)(S-\gamma)(S-\delta)=$$

$$(2-\alpha)(2-\beta)(2-\gamma)(2-\delta)=$$

$$16-8(\alpha+\beta+\gamma+\delta)+4(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)-2(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+\alpha\beta\gamma\delta=$$

$$=16-8(2)+4(3)-2(5)+7$$

$$=9$$

(b) Note: Since some of the algebra for this problem takes up a great deal of space, some steps have been skipped for brevity's sake. Readers are encouraged to verify each step on their own.

Let $g(x^2)=(x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)=f(x)f(-x)$, since each of the terms factors and the new roots of this equation are just the negative of the roots of the previous function *f(x)*.Using the same logic as the previous part, we desire to find is $g(\alpha^2+\beta^2+\gamma^2+\delta^2)$.

Now, let's calculate *g(x²):*

$$g(x^2) = f(x)f(-x) = (x^4 - 2x^3 + 3x^2 - 5x + 7)(x^4 + 2x^3 + 3x^2 + 5x + 7)$$

Due to the symmetry of the factors, all the odd terms cancel out and the polynomial is left with only even terms. Multiplying out we find:

$$g(x^2) = x^8 + 2x^6 + 3x^4 + 17x^2 + 49$$

Thus, $g(x) = x^4 + 2x^3 + 3x^2 + 17x^1 + 49$. Now, we just simply determine $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. Take the first original equation given and square both sides:

$$(\alpha + \beta + \gamma + \delta)^2 = 2^2$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = 4$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(3) = 4$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$$

It follows that the quantity we desire is $g(-2) = 16 - 16 + 12 - 34 + 49 = 27$.

Another alternate solution that is less elegant than the one above:

Since $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$, we find that the quantity we desire to find can be expressed as:

$$(-2 - \alpha^2)(-2 - \beta^2)(-2 - \gamma^2)(-2 - \delta^2) =$$
$$(2 + \alpha^2)(2 + \beta^2)(2 + \gamma^2)(2 + \delta^2) =$$

$$16 + 8(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + 4((\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2) +$$
$$2((\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2) + \alpha^2\beta^2\gamma^2\delta^2 =$$

$$16 + 8(-2) + 4((\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2) +$$
$$2((\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2) + 7^2 =$$

$$= 49 + 4((\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2)$$
$$+ 2((\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2)$$

At this point, we are left with finding the two following quantities:

$$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2$$

$$(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$$

Let X = $(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2$, and
let Y = $(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$. We seek to find $49 + 4X + 2Y$.

To solve for Y, take the third equation originally given and square it:

$$(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)^2 = 5^2$$

$$(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2 + 2(\alpha\beta)^2\gamma\delta + 2(\alpha\gamma)^2\beta\delta + 2(\alpha\delta)^2\beta\gamma + 2(\beta\gamma)^2\alpha\gamma + 2(\beta\delta)^2\alpha\gamma + 2(\gamma\delta)^2\,\alpha\beta = 25$$

$$Y + 2\alpha\beta\gamma\delta(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = 25$$

$$Y + 2(7)(3) = 25$$

$$Y = -17$$

Now, we solve for X. Take the second equation originally given and square it:

$$(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^2 = 3^2$$

$$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\alpha\delta)^2 + (\beta\gamma)^2 + (\beta\delta)^2 + (\gamma\delta)^2 + 2\alpha^2\beta\gamma + 2\alpha^2\beta\delta + 2\alpha^2\gamma\delta + 2\alpha\beta^2\gamma + 2\alpha\beta^2\delta + 2\beta^2\gamma\delta + 2\alpha\beta\gamma^2 + 2\alpha\gamma^2\delta + 2\beta\gamma^2\delta + 2\alpha\beta\delta^2 + 2\alpha\gamma\delta^2 + 2\beta\gamma\delta^2 + 6\alpha\beta\gamma\delta = 9$$

$$X + 2\alpha\beta\gamma(\alpha + \beta + \gamma) + 2\alpha\beta\delta(\alpha + \beta + \delta) + 2\alpha\gamma\delta(\alpha + \gamma + \delta) + 2\beta\gamma\delta(\beta + \gamma + \delta) + 6\alpha\beta\gamma\delta = 9$$

$$X + 2\alpha\beta\gamma(2 - \delta) + 2\alpha\beta\delta(2 - \gamma) + 2\alpha\gamma\delta(2 - \beta) + 2\beta\gamma\delta(2 - \alpha) + 6(\alpha\beta\gamma\delta) = 9$$

$$X + 4(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) - 8\alpha\beta\gamma\delta + 6\alpha\beta\gamma\delta = 9$$
$$X + 4(5) - 2(7) = 9$$
$$X = 3$$

It follows that the quantity we desire, $49 + 4X + 2Y = 49 + 4(3) + 2(-17) = 27$.

3) How many positive odd integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 200?$$

Solution

Let's add a dummy variable, x_7 to the end of the left hand side of the inequality and let it equal the non-negative "slack" in between the sum of the first six numbers and 200. Thus, we are now looking for solutions in the following equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 200$$

where x_1, x_2, x_3, x_4, x_5, and x_6, are odd. Note that each solution for this equation maps uniquely to a solution to the previously given inequality by removing x_7 from the solution.

Since we know that these first six variables are odd, we can create new variables y_i, $(1 \leq i \leq 6)$, with $x_i = 2y_i + 1$, for non-negative integers y_i. Substituting we must find the total number of non-negative solutions to the equation:

$$2y_1 + 1 + 2y_2 + 1 + 2y_3 + 1 + 2y_4 + 1 + 2y_5 + 1 + 2y_6 + 1 + x_7 = 200$$
$$2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + x_7 = 194$$

Notice that for this equation to have any solutions, x_7 must also be even. Thus, we can also substitute $x_7 = 2y_7$, where y_7 is a non-negative integer:

$$2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 = 194$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 = 97$$

Thus, the answer to the original question is equal to the number of non-negative integer solutions to the equation above. This is a combinations with repetition problem, where we are looking for the number of combinations of size 97 chosen from 7 distinct items. There are $\binom{97+7-1}{7-1} = \binom{103}{6}$ of these.

4) There are *n* teams in a round robin tournament. Each team plays every other team exactly once. In each match the chance of each team winning is exactly 50%. What is the probability that there is neither an undefeated team nor a winless team at the end of the tournament?

Solution

There are a total of $\binom{n}{2}$ games played in the tournament, so the total number of possible outcomes, the sample space of the problem, is $2^{\binom{n}{2}}$. To obtain the desired probability, we must count the number of these outcomes that correspond to no undefeated and no winless team.

It will be easier to count the opposite: the number of outcomes that have either at least one undefeated team or one winless team. First, note that no tournament where every pair of teams plays each other can produce either two undefeated teams or two winless teams, since every pair of teams plays against each other and in any match both teams can't win and both teams can't lose. But, a tournament may have exactly one undefeated team and one winless team. Thus, we must use the inclusion-exclusion principle to count the number of outcomes with either a undefeated or winless team.

There are *n* distinct teams, each of which could go undefeated. If we set one team's games all to wins, there are $\binom{n}{2} - n$ other games, each of which can have any outcome while the tournament has an undefeated team. Thus, the total number of tournaments with an undefeated team is $n2^{\binom{n}{2}-n}$.

This same exact logic works to determine the number of tournaments with winless teams. But, in this count, we have included some tournaments, namely those with both undefeated and winless teams. We must subtract these out of our final count. There are *n* ways in which we can choose an undefeated team and *n-1* ways in which we can choose a winless team. For each of these choices, we are fixing *2n-1* games, *n* of the winner's games to be wins, and the other *n-1* games of the winless team to be losses. (Note in the first team winning all of its games, the winless team was already assigned a loss.) Thus, the outcomes of the remaining $\binom{n}{2} - (2n-1)$ games are free to be either wins or losses. It follows that there are $n(n-1)2^{\binom{n}{2}-(2n-1)}$ ways in which we get tournaments with both an undefeated team and a winless team.

Applying the Inclusion-Exclusion principle, we find a total of

$$2n2^{\binom{n}{2}-n} - n(n-1)2^{\binom{n}{2}-(2n-1)}$$

possible outcomes with either an undefeated or winless team. Subtracting this from the total we find

$$2^{\binom{n}{2}} - 2n2^{\binom{n}{2}-n} + n(n-1)2^{\binom{n}{2}-(2n-1)}$$

outcomes where there is no undefeated or winless team. Thus, the desired probability is:

$$\frac{2^{\binom{n}{2}} - 2n2^{\binom{n}{2}-n} + n(n-1)2^{\binom{n}{2}-(2n-1)}}{2^{\binom{n}{2}}}$$

Alternative, we can simplify this to be:

$$1 - \frac{n}{2^{n-1}} + \frac{n(n-1)}{2^{2n-1}}$$

5) Let *a, b, c* be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

<u>Solution</u>

The Arithmetic-Harmonic Mean Inequality states that the arithmetic mean of a set of positive real numbers is greater than or equal to the harmonic mean of the same numbers.(http://www.artofproblemsolving.com/Wiki/index.php/Root-Mean_Square-Arithmetic_Mean-Geometric_Mean-Harmonic_mean_Inequality) It follows that the reciprocal of the harmonic mean is greater than the reciprocal of the arithmetic mean of the same set of positive real numbers. Specifically for three positive real numbers *x, y, z*, we have:

$$\frac{1}{\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}}\geq\frac{1}{\frac{x+y+z}{3}}$$

Simplifying this, we find that for all positive real numbers *x, y, z*:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{9}{x+y+z}$$

Now, let's work with the quantity on the left hand side of the inequality:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=$$
$$\frac{a+b+c-(b+c)}{b+c}+\frac{a+b+c-(a+c)}{c+a}+\frac{a+b+c-(a+b)}{a+b}=$$
$$\frac{a+b+c}{b+c}-1+\frac{a+b+c}{c+a}-1+\frac{a+b+c}{a+b}-1=$$
$$(a+b+c)(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b})-3=$$

Now, plug in *x*=*b*+*c*, *y*=*c*+*a*, *z*=*a*+*b* into the inequality previously derived to yield:

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq\frac{9}{b+c+c+a+a+b}=\frac{9}{2(a+b+c)}$$

Getting back to our problem, we now have that

$$(a+b+c)(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b})-3\geq$$
$$(a+b+c)\left(\frac{9}{2(a+b+c)}\right)-3=$$
$$\frac{9}{2}-3=\frac{3}{2}$$

Proving the given assertion.