University of Central Florida
Online Mathematics Contest
Solutions: March 2015 (Year 2, Round 4)

<u>Warm Ups</u>

1) If $log_x y = 2log_z y$ what is $log_x z$? Assume that x, y and z are all greater than 1.

<u>Solution</u>
We simplify the equation as follows: $\frac{\ln y}{\ln x} = \frac{2lny}{lnz}$. Since y ≠ 0, we can cancel these terms in the numerator and obtain $\frac{lnz}{lnx} = 2$, which we can rewrite as $log_x z = 2$. If any of x, y and z are equal to 1, then the question isn't well-defined.

2) Given that $z = -2 + 7i$ is a root to the equation: $z^3 + 6z^2 + 61z + 106 = 0$, what are the other two roots of the equation?

<u>Solution</u>
Since the coefficients are all real, it follows that the complex conjugate of $-2 + 7i$ is also a root of the equation. Thus, two of the roots are $-2 + 7i$ and $-2 - 7i$. Since the sum of the roots is -6, it follows that the last root is -2.

3) Find all the points of intersections of the circle $x^2 + 2x + y^2 + 4y = -1$ and the line $x - y = 1$.

<u>Solution</u>
Plug in $x = y + 1$ into the equation of the circle:

$$(y+1)^2 + 2(y+1) + y^2 + 4y = -1$$
$$y^2 + 2y + 1 + 2y + 2 + y^2 + 4y = -1$$
$$2y^2 + 8y + 4 = 0$$
$$y^2 + 4y + 2 = 0$$
$$y = \frac{-4 \pm \sqrt{16 - 4(2)}}{2} = -2 \pm \sqrt{2}$$

Rewriting the circle equation in standard form, we get

$$x^2 + 2x + 1 + y^2 + 4y + 4 = -1 + 5$$
$$(x+1)^2 + (y+2)^2 = 2^2$$

Plugging in either value of y, we get:

$$(x+1)^2 + (-2 \pm \sqrt{2} + 2)^2 = 2^2$$
$$(x+1)^2 + (\pm\sqrt{2})^2 = 2^2$$

$$(x+1)^2+2=2^2$$
$$(x+1)^2=2$$
$$x+1=\pm\sqrt{2}$$
$$x=-1\pm\sqrt{2}$$

We must check with the line equation to verify that the two valid points of intersection are $(-1-\sqrt{2},-2-\sqrt{2})$ and $(-1+\sqrt{2},-2+\sqrt{2})$.

4) Rubba and Prince decided that they were going to have a contest who could get the most money in March. Each one of them came up with a different way to beat the other. At the end of the month, Rubba ended up doing several odd jobs around Orlando and made $10 million dollars. Prince, on the other hand made a deal with the university that he should get a penny one day, and double the amount of pennies the next day, and then double the pennies of previous day, continually for each day in March. Who made the most money in March?

Solution

The amount Price received is $\sum_{i=1}^{31}(\$.01)2^{i-1}=(\$.01)\sum_{i=0}^{30}2^i=(\$.01)(2^{31}-1)=$ $\$21474836.74$. This is a bit more than twice what Rubba received. Thus, Prince made the most money in March.

5) Alaina's age uses the same two digits as Brett's age, but the two digits are in reverse order in her age. The difference between the square of Alaina and Brett's ages is also a perfect square. If Alaina is older than Brett, how old are they both?

Solution

Let the two digits be x and y with Alaina being 10x + y years old and Brett being 10y + x years old. (Note that since Alaina is older, we have x > y.) Using the given information, for some positive integer n, we have:

$$(10x+y)^2-(10y+x)^2=n^2$$
$$100x^2+20xy+y^2-100y^2-20xy-x^2=n^2$$
$$99x^2-99y^2=n^2$$
$$99(x^2-y^2)=n^2$$

It follows that n^2 is divisible by 99, implying that n is divisible by 33, each each unique prime factor in 99 must appear at least once in n. n must be strictly less than 99, since n is a perfect square less than the square of Alaina's age, thus, n = 33 or n = 66. Substituting n = 33, we find:

$$x^2-y^2=11$$

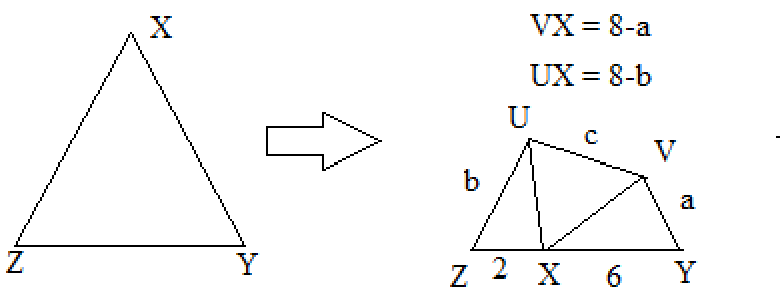
Two digits that satisfy this equation are x = 6 and y = 5. If we substitute n = 66, get a difference of squares of digits equal to 44, for which there is no solution. Thus, Alaina is 65 and Brett is 56.

Exercises

1) William is making origami and currently has a paper equilateral triangle, XYZ with side length 8. He folds the triangle so that vertex X touches a point on the side YZ, with a distance 6 from point Y. What is the length of the fold?

Solution

Here is a drawing of the triangle unfolded and folded:



Recall that the original triangle is equilateral, so that the angles at X, Y and Z are each 60º. Using the law of cosines, we set up an equation based on triangle VXY to solve for a:

$$(8-a)^2 = 6^2 + a^2 - 2(6)(a)\cos(60°)$$
$$64 - 16a + a^2 = 36 + a^2 - 12a(\frac{1}{2})$$
$$10a = 28$$
$$a = \frac{14}{5}$$

Using the law of cosines, we set up an equation based on triangle UZX to solve for b:

$$(8-b)^2 = 2^2 + b^2 - 2(2)(b)\cos(60°)$$
$$64 - 16b + b^2 = 4 + b^2 - 4b(\frac{1}{2})$$
$$14b = 60$$
$$b = \frac{30}{7}$$

Now, we can use the law of cosines on triangle UXV to solve for c, the length of the fold:

$$c^2 = (\frac{26}{5})^2 + (\frac{26}{7})^2 - 2(\frac{26}{5})(\frac{26}{7})\cos(60°)$$
$$c^2 = 26^2(\frac{1}{25} + \frac{1}{49} - \frac{1}{35})$$
$$c^2 = 26^2(\frac{49 + 25 - 35}{25 \times 49})$$
$$c = \frac{26}{35}\sqrt{39} \sim 4.64$$

2) A number with 3 digits in consecutive ascending order is multiplied by another 3-digit number with digits in consecutive descending order. The product of these two numbers is 110745. What are the two numbers?

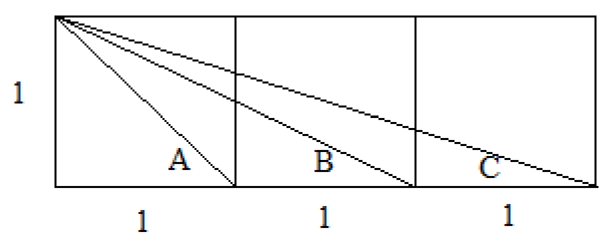
Solution

Let the first digit in the first number be a and the first digit in the last number be b. Then, the two numbers are:

100a + 10(a+1) + (a+2)
100b + 10(b-1) + (b-2)

If we isolate the last digit in the product, we get the last digit of (a + 2)(b - 2), which must be 5. Since 5 is prime, it follows that either a + 2 = 5 or b - 2 = 5. If the latter were the case, then b = 7 and the second number would be 765, but this isn't possible because 110745 isn't divisible by 765. It follows that a = 3 and the first number is 345. Since 110745/345 = 321, it follows that b = 3 also. Thus, the two numbers are 345 and 321.

3) Consider the diagram with three unit squares side by side below:



where A, B and C are the acute angles denoted. What is the sum of angles A, B and C?

Solution

We have $tanA = 1$, $tanB = \frac{1}{2}$, and $tanC = \frac{1}{3}$.

$$\tan(A+B) = \frac{tanA + tanB}{1 - tanAtanB} = \frac{1 + {}^1\!/_2}{1 - {}^1\!/_2} = \frac{{}^3\!/_2}{{}^1\!/_2} = 3$$

Now, note that by definition of tangent in a right triangle $tanA = \frac{1}{\tan(90° - A)}$. To see this, consider a right triangle ABC with a right angle at C. By definition $tanA = \frac{BC}{AC}$. Likewise, for angle B we find. $tanB = \tan(90° - A) = \frac{AC}{BC}$. Since 3 and $\frac{1}{3}$ are reciprocals and both angles A+B and C are acute, it follows that the sum of the angles, (A+B)+C, must be 90º.

4) Without using a calculator, determine with proof, the exact value of

$$(1 + tan20°)(1 + tan25°).$$

Solution

Note that $1 = tan45° = \tan(20° + 25°) = \frac{tan20° + tan25°}{1 - tan20° tan25°}$.
It follows that $tan20° + tan25° = 1 - tan20° tan25°$.

Now, let's expand the given expression:

$$\begin{aligned}(1 + tan20°)(1 + tan25°) &= 1 + tan20° + tan25° + tan20° tan25° \\ &= 1 + (1 - tan20° tan25°) + tan20° tan25° \\ &= 2\end{aligned}$$

5) Prove that $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$, for all positive integers n. (Note: You may find a combinatorial argument more elegant than an algebraic proof.)

Solution

This is a standard problem in beginning combinatorics, typically proven with a combinatorial argument. Consider n distinct red balls and n distinct blue balls in a bin. We can choose exactly n of these balls in $\binom{2n}{n}$, by definition of a combination. Now, consider counting these combinations in a different way. Split up the counting into n+1 groups: combinations with 0 red balls, 1 red ball, 2 red balls, ..., and n red balls. In general, to count the number of combinations with *i* red balls, note that we must choose *i* red balls out of n and combine this choice with a choice of *n-i* blue balls out of *n* blue balls. Thus, the number of combinations with *i* red balls is $\binom{n}{i}\binom{n}{n-i}$, since each choice of red balls can be paired with each choice of blue balls. Summing over each possible number of red balls, we have the total number of combinations equal to $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$. Finally, note that the number of ways to choose *i* balls out of *n*, is equal to the number of ways **NOT** to choose *n-i* balls out of *n*. Namely, there is a one-to-one correspondence between each choice of *i* balls out of *n* with the other *n-i* balls. Thus, we have the identity $\binom{n}{i} = \binom{n}{n-i}$. Thus, in breaking our counting down into different groups based on the number of red balls in each combination, we get $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2$ showing the desired equivalence.

Investigations

1) Let S be a subset of {1, 2, 3, 4, ..., 50} such that the difference between each pair of items in S is distinct. With proof, what is the largest possible size of S? Give a set of this size that satisfies the requirement. (Note: the set {1, 3, 7, 15} satisfies the difference requirement for S since the 3 - 1 = 2, 7 - 1 = 6, 15 - 1 = 14, 7 - 3 = 4, 15 - 3 = 12 and 15 - 7 = 8, and all of these differences, 2, 6, 14, 4, 12 and 8 are all different.)

Solution

First note that we can compose a set S with 9 elements: {1, 2, 4, 11, 17, 22, 36, 44, 48}. The corresponding 36 pairwise differences are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 25, 26, 27, 31, 32, 33, 34, 35, 37, 40, 42, 43, 44, 46, and 47. (Note that there are 562 valid sets of size 9 that exist. This set was the first set in lexicographical order generated with the attached computer program investigation1.java.)

Assume to the contrary that we could compose a set S with 10 elements. Let these elements in sorted order be $a_1, a_2, ... a_{10}$. For 1 ≤ i ≤ 9, let $d_i = a_{i+1} - a_i$, and for 1 ≤ i ≤ 8, let $e_i = a_{i+2} - a_i$. Note that these 17 values which we've defined must all be distinct and positive. Thus, we can set up the equation:

$$\sum_{i=1}^{9} d_i + \sum_{i=1}^{8} e_i \geq \sum_{i=1}^{17} i = 153$$

However, note that the sum of the differences of adjacent terms ($\sum_{i=1}^{9} d_i$) must be less than 49, since adding all of these equals $a_{10} - a_1 \leq 50 - 1 = 49$. Also note that $\sum_{i=1}^{8} e_i = \sum_{i=1}^{8}(d_i + d_{i+1}) \leq \sum_{i=1}^{9}(d_i + d_i) = \sum_{i=1}^{9} 2d_1 \leq 2(49) = 98$.

Thus, using these two minimizing constraints, we find

$$\sum_{i=1}^{9} d_i + \sum_{i=1}^{8} e_i \leq (50-1) + 2(50-1) = 147$$

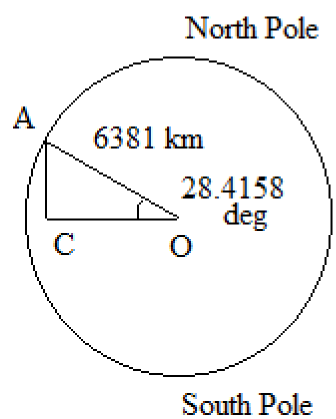
Clearly, the solution set to these two inequalities is empty. It follows that no set of values be $a_1, a_2, ... a_{10}$ exists that satisfies the given constraints. Coupled with the example of a set S with size 9, this proves that 9 is the maximal size of any possible set S that satisfies the given constraint.

2) Orlando, Florida (USA) is located at 28.4158º N latitude and 81.2989º W longitude (according to Google) while Melbourne, Australia is located at 37.8136º S latitude and 144.9631º E longitude (according to the same source). Assuming that the Earth is a sphere with radius 6371 kilometers, what's the length of the shortest possible plane flight between the two cities, assuming that the plane stays 10 km above the ground for the whole trip. For simplicity, just count the distance once the plane has elevated 10 km above the ground. Assume that the plane rises to this elevation at the two latitude/longitude pairs given in the problem.

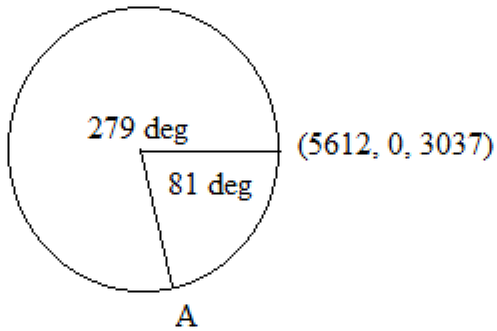
Solution

Let O be the center of the earth, A be the point 10 km above the ground in Orlando and B be the point 10 km above the ground in Melbourne. If we can find the angle AOB, then we can simply use the arc length formula to find the plane flight. The tool best equipped for this task is vectors. let O have the coordinates (0, 0, 0) in km. Let the point on the earth at 0º longitude and 0º latitude be (6371, 0, 0) in km and the point on earth at 90º E longitude and 0º latitude be (0, 6371, 0). Finally, let the point at 90º N latitude be (0, 0, 6371).

First, let's calculate the (x, y, z) coordinates of the point A. We can first calculate the correct z coordinate of this point. We simply form a right triangle with an angle of 28.4158º and a hypotenuse of 6381 km. The side opposite the angle represents the z-coordinate. Here is a picture where the vertical is the z axis.



Thus, the desired z coordinate is 6381sin(28.4158º) ~ 3036.505825. CO = 6381cos(28.4158º) = 5612.200404. This latter value helps us because all of the points on the sphere that are at the z coordinate 3036.505825 form a circle of radius 5612.200404. We can get the desired x and y coordinates by finding the point on the circle at θ = 360º - 81.2989º = 278.7011º. This point (including our z coordinate) is (5612.200404cosθ, 5612.200404sinθ, 3036.505825). Simplifying, we get (849.0125431, -5547.609492, 3036.505825). Here is a diagram taken from the perspective of the North Pole (latitude 90º N) of the horizontal plane containing A. Note that the numbers have been approximated so they fit easily:



Now, we use the same process to get the coordinates of B. The z-coordinate for B is 6381sin(-37.8136º) ~ -3912.156587. The corresponding radius of the circle containing the horizontal plane containing B is 6381cos(37.8136º) = 5041.050668. It follows that (x, y, z) coordinates of Melbourne, Austrailia are (5041.050668cos144.9631º, 5041.050668sin144.9631º, -3912.156587) which is roughly (-4127.523947, 2894.086712, -3912.156587).

To recap, we desire the great arc distance between the two points:

(849.0125431, -5547.609492, 3036.505825) and

(-4127.523947, 2894.086712, -3912.156587).

These points also represent the two vectors to those points from the origin, O. Let these vectors be v_1 and v_2, respectively. We can use the dot product of the two vectors to calculate the angle AOB:

$$v_1 \circ v_2 = 849.0125431 \times -4127.523947 + (-5547.609492) \times 2894.086712 + 3036.505825 \times (-3912.156587) \sim -31438868.78$$

$$v_1 \circ v_2 = 6183 \times 6183 \times \cos(AOB) = 38229489\cos(AOB)$$

$$v_1 \circ v_2 = 38229489 \cos(AOB) = -31438868.78$$
$$\cos(AOB) \sim -.8223721949$$
$$AOB \sim 145.3229695°$$

Thus, the distance flow is $6381km \times \frac{145.3229695°}{360°} \times 2\pi = 16184.54057km$.

Rounded to the nearest kilometer, the flight is 16185 km.

3) A bracket for a single elimination tournament between eight seeded teams looks like this:

___(1)___				___(2)___
vs	___(W 1v8)___		___(W 2v7)___	vs
___(8)___				___(7)___
	vs.	_(W left)_ vs _(W right)_		
___(4)___				___(3)___
vs	___(W 4v5)___		___(W 3v6)___	vs
___(5)___				___(6)___

Seeds represent pre-rankings for teams with the first seed being the best team and the eighth seed being the worst team. The four first round games are denoted on the left and right ends of the bracket. The winners of the two games on each side of the bracket then play each other. Finally the winners of those two games play each other for the championship. For a given tournament assume that when the i^{th} seeded team plays the j^{th} seeded team, the chance of the i^{th} seeded team winning is $\frac{j}{i+j}$. What is the probability that the first seeded team wins the whole tournament?

<u>Solution</u>

The chance of the first seeded team winning the first game is $\frac{8}{9}$. In $\frac{4}{9}$ of those cases it would play the 5th ranked team and in $\frac{5}{9}$ of those cases it would play the 4th ranked team. Thus, the probability that the first ranked team advances to the finals is

$$\frac{8}{9}\times\left(\frac{4}{9}\times\frac{5}{6}+\frac{5}{9}\times\frac{4}{5}\right)=\frac{176}{243}$$

Now, we must work out the probability that each of the teams on the opposite side of the bracket make it to the finals.

Using the framework shown above, we calculate the probability that the second team makes the finals as

$$\frac{7}{9}\times\left(\frac{3}{9}\times\frac{6}{8}+\frac{6}{9}\times\frac{3}{5}\right)=\frac{91}{180}$$

The probability the 3rd ranked team makes the finals is:

$$\frac{6}{9}\times\left(\frac{2}{9}\times\frac{7}{10}+\frac{7}{9}\times\frac{2}{5}\right)=\frac{14}{45}$$

The probability that the 6th ranked team makes the finals is:

$$\frac{3}{9}\times\left(\frac{2}{9}\times\frac{7}{13}+\frac{7}{9}\times\frac{2}{8}\right)=\frac{49}{468}$$

The probability the 7th ranked team makes the finals is:

$$\frac{2}{9} \times \left(\frac{3}{9} \times \frac{6}{13} + \frac{6}{9} \times \frac{3}{10}\right) = \frac{46}{585}$$

Now, given that the first ranked team is in the finals, its chance of winning is

$$\frac{91}{180} \times \frac{2}{3} + \frac{14}{45} \times \frac{3}{4} + \frac{49}{468} \times \frac{6}{7} + \frac{46}{585} \times \frac{7}{8} = \frac{5117}{7020}$$

Finally, we must multiply this by the probability of the first place team getting to the finals, to calculate the first seeded team's chance of winning, which is:

$$\frac{176}{243} \times \frac{5117}{7020} = \frac{225148}{426465} \sim .528$$

4) With proof, determine the value of the sum $\sum_{i=1}^{n}(-1)^{i+1} i^2$, in terms of n.

<u>Solution</u>

Using induction on n, we'll show $\sum_{i=1}^{n}(-1)^{i+1} i^2 = (-1)^{n+1}\frac{n(n+1)}{2}$, for all positive integers n.

Base case, n = 1. The LHS = $(-1)^{1+1}1^2 = 1$, RHS = $(-1)^{1+1}\frac{1(1+1)}{2} = 1$, thus the equation is true for n = 1.

Inductive hypothesis: Assume for an arbitrary value of n = k that the equation holds. Namely, that $\sum_{i=1}^{k}(-1)^{i+1} i^2 = (-1)^{k+1}\frac{k(k+1)}{2}$.

Inductive step: Under this assumption, we must prove for n = k+1 that $\sum_{i=1}^{k+1}(-1)^{i+1} i^2 = (-1)^{k+2}\frac{(k+1)(k+2)}{2}$.

$$\begin{aligned}\sum_{i=1}^{k+1}(-1)^{i+1} i^2 &= \left(\sum_{i=1}^{k}(-1)^{i+1} i^2\right) + (-1)^{k+2}(k+1)^2 \\ &= (-1)^{k+1}\frac{k(k+1)}{2} + (-1)^{k+2}(k+1)^2 \\ &= (-1)^{k+2}[-\frac{k(k+1)}{2} + (k+1)(k+1)] \\ &= (-1)^{k+2}(k+1)[-\frac{k}{2} + \frac{2(k+1)}{2}] \\ &= (-1)^{k+2}(k+1)[\frac{2k+2-k}{2}] \\ &= (-1)^{k+2}\frac{(k+1)(k+2)}{2}\end{aligned}$$

This proves the inductive step. Thus, we can conclude that the formula is true for all positive integers n.

5) Prove that $\prod_{i=1}^{n} \cos\left(\frac{\alpha}{2^i}\right) = \frac{sin\alpha}{2^n \sin(\frac{\alpha}{2^n})}$, for all positive integers n and positive acute angles α.

Solution

Once again, we use induction on n.

Base case: n = 1, LHS = $\prod_{i=1}^{1} \cos\left(\frac{\alpha}{2^i}\right) = \cos(\frac{\alpha}{2})$, RHS = $\frac{sin\alpha}{2^1 \sin(\frac{\alpha}{2^1})} = \frac{2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2})}{2\sin(\frac{\alpha}{2})} = \cos(\frac{\alpha}{2})$.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\prod_{i=1}^{k} \cos\left(\frac{\alpha}{2^i}\right) = \frac{sin\alpha}{2^k \sin(\frac{\alpha}{2^k})}$$

Inductive step: Prove for n = k+1 that $\prod_{i=1}^{k+1} \cos\left(\frac{\alpha}{2^i}\right) = \frac{sin\alpha}{2^{k+1} \sin(\frac{\alpha}{2^{k+1}})}$

$$\begin{aligned}\prod_{i=1}^{k+1} cos\left(\frac{\alpha}{2^i}\right) &= \left(\prod_{i=1}^{k} \cos(\frac{\alpha}{2^i})\right)\cos(\frac{\alpha}{2^{k+1}}) \\ &= \frac{sin\alpha}{2^k \sin(\frac{\alpha}{2^k})} \times \cos(\frac{\alpha}{2^{k+1}}) \\ &= \frac{sin\alpha}{2^k 2\sin(\frac{\alpha}{2^{k+1}})\cos(\frac{\alpha}{2^{k+1}})} \times \cos(\frac{\alpha}{2^{k+1}}) \\ &= \frac{sin\alpha}{2^{k+1}\sin(\frac{\alpha}{2^{k+1}})}\end{aligned}$$

This proves the inductive step. We can conclude that the given equation is true for all positive integers n.