University of Central Florida
Online Mathematics Contest
Solutions: November 2014 (Year 2, Round 1)

Warm Ups

1) John takes 75 minutes to mow one acre and Sally takes 60 minutes to mow one acre. How many minutes would it take them working together to mow a 6 acre lot?

Solution

Let t be the number of minutes both John and Sally mow the 6 acre lot. It follows that John would mow $\frac{t}{75}$ acres and Sally would mow 60 acres. Since they mow 6 acres together, we have:

$$\frac{t}{75} + \frac{t}{60} = 6$$
$$\frac{4t}{300} + \frac{5t}{300} = 6$$
$$9t = 6(300)$$
$$t = 200$$

Thus, it would take them 200 minutes (3 hours and 20 minutes) to mow the six acre lot, working together.

2) A class has n students in it. Their average test grade was 78. When Alice's grade is removed from the group, the remaining students had an average test grade of 73. What is the maximum value of n for which this information is plausible? For this value of n, what must Alice's test score be?

Solution

The sum of the all of the students' scores is 78n. The sum of the scores without Alice's test is 73(n - 1), since there are n - 1 students remaining with an average score of 73. Let A be Alice's test score. This gives us the following equation:

$$73(n-1) + A = 78n$$
$$73n - 73 + A = 78n$$
$$A = 5n + 73$$

If we assume that the maximum test score is 100, then we find that the largest integer n that allows for $A \leq 100$ is $n = 5$. For this value of *n*, Alice scored *5(5) + 73 = 98.*

3) The 15th term in an arithmetic sequence is 68. If the sum of the first twenty terms of the sequence is 1000, what is the value of the first term of the sequence?

Solution

Let the sequence be denoted as a_1, a_2, ..., a_{20} with a common difference of d. Using the given information, we have:

$$a_{15} = 68, \frac{(a_1 + a_{20})20}{2} = 1000$$

Thus, we find that $a_1 + a_{20} = 100$, simplifying the second equation. Now, utilizing the common difference, we find that

$$a_1 = a_{15} - 14d$$
$$a_{20} = a_{15} + 5d$$

Adding these two equations we find:

$$a_1 + a_{20} = a_{15} - 14d + a_{15} + 5d$$
$$100 = 2a_{15} - 9d$$
$$100 = 2(68) - 9d$$
$$9d = 36$$
$$d = 4$$

It follows that $a_1 = a_{15} - 14d = 68 - 14(4) = 12$.

4) Consider writing the positive integers in increasing order. What would be the 2014th digit written? (For example, the 20th digit written would be 1, since there are 9 digits in 1 – 9 and 10 digits in 10 – 14, so the 20th digit would be the 1 in writing 15.)

Solution

The first nine digits correspond to 1 - 9. The next 180 digits correspond to 10 - 99. Thus, we are looking for the 2014 - 9 - 180 = 1825th digit written when we start with writing 100. Since these numbers each have three digits, we calculate that 1825/3 = 608, using integer division. This means that we will write 1824 digits writing the numbers 100 through 707. The 1825th digit we write starting at 100 is 7, the first digit while writing 708. Thus, 7 is the 2014th digit written overall.

5) What is the value of $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$?

Solution

Let x equal the quantity in question.

$$x = \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$$

$$x^2 = 6 + \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$$

$$x^2 = 6 + x$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

Since we know that x is positive, we can conclude that x must be 3.

Exercises

1) How many positive integers are divisors of exactly two of the three numbers 15^7, 18^5 and 20^6?

Solution

First, let's prime factorize the three numbers given:

$$15^7 = 3^7 5^7,\ 18^5 = 2^5 3^{10},\text{ and } 20^6 = 2^{12} 5^6$$

Notice that no pair of these numbers share more than one prime factor in common. Thus, the only positive integers that are divisors of exactly two of these numbers are individual primes raised to a power. For example, all the values of the form 3^a with $1 \le a \le 7$ are divisors of 15^7 and 18^5, but not 20^6. There are precisely 7 of these. Similarly, all numbers of the form 2^b with $1 \le b \le 5$ are divisors of 18^5 and 20^6 but not 15^7. Finally all the values of the form 5^c with $1 \le c \le 6$ are divisors of 15^7 and 20^6 but not 18^5. Adding these up, we have 18 integers that are divisors of exactly two of the given three numbers.

2) How many zeroes are at the end of 2014!?

Solution

This is equivalent to asking, "what is the highest power of 10 that divides evenly into 2014!"? To answer this question, we simply need to know the highest power of 2 and 5 that divide evenly into 2014!. Let's try to solve a more general problem and then apply our solution to this specific instance:

What is the highest power of a prime number p that divides evenly into n!?

Imagine writing out n! = 1 x 2 x 3 x ... x n.

We can cross off every p^{th} value and add one to our count of times p divides into n! Since we are starting with the p^{th} value (and not the first), the number of values we cross off is $\left\lfloor \frac{n}{p} \right\rfloor$. But, we're not done. What if p^2 is on the list? Then, we would have only crossed off 1 factor of p for this term, but in reality it has another factor of p. In fact, for all multiples of p^2, we only crossed off 1 of 2 (or more) possible multiples. Thus, we want to make a second pass through our values, crossing off one more copy of p from each multiple of p^2, of which there are $\left\lfloor \frac{n}{p^2} \right\rfloor$. But of course, if p^3 were on the list, we would have only crossed off 2 of its copies of p. This argument continues for k iterations, where $p^k \le n \le p^{k+1}$. Mathematically, we can express our answer most simply using an infinite summation, since the latter terms will all be 0. Thus, the number of times a prime p divides into n! is:

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

Applying our result to the specific query, we can run this calculation twice, once with n = 2014 and p = 2 and another time with n = 2014 and p = 5. Of the two results, we want to take the minimum, since each copy of 10 has one copy of 2 and one copy of 5. It should be fairly clear to see that increasing p will reduce the sum, thus, it suffices to determine the number of times 5 divides evenly into 2014! Manually calculating the sum we get:

$$\left\lfloor \frac{2014}{5} \right\rfloor + \left\lfloor \frac{2014}{25} \right\rfloor + \left\lfloor \frac{2014}{125} \right\rfloor + \left\lfloor \frac{2014}{625} \right\rfloor = 402 + 80 + 16 + 3 = 501$$

3) Determine the following sum in terms of n: $\sum_{i=1}^{2^n} \lfloor log_2 i \rfloor$. Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Solution

Notice that each term in the sum is an integer ranging from 0 to n, inclusive. (Only the first term is 0 and only the last term is n. The rest are in between 1 and n - 1, inclusive.)

Thus, we must determine the number of times each of these terms appears in the sum.

The value k appears in the sum for each value of i ranging from 2^k to $2^{k+1} - 1$, since $\log_2 2^k = k$ and $\log_2 2^{k+1} = k + 1$.

There are precisely $2^{k+1} - 1 - 2^k + 1 = 2^{k+1} - 2^k = 2^k$ values in this range. Thus, an equivalent expression to the one above is

$$n + \sum_{k=1}^{n-1} k2^k$$

Let's work on the summation portion of this expression. Let this sum be S.

$$S = 1(2^1) + 2(2^2) + 3(2^3) + 4(2^4) + \cdots + (n-1)(2^{n-1}).$$

Multiply this expression through by 2:

$$2S = 1(2^2) + 2(2^3) + 3(2^4) + 4(2^5) + \cdots + (n-2)(2^{n-1}) + (n-1)(2^n)$$

Now, let's subtract the bottom equation from the top, notice the terms realigned:

$$\begin{array}{rl} S = & 1(2^1) + 2(2^2) + 3(2^3) + 4(2^4) + \cdots + (n-1)(2^{n-1}). \\ -2S = & \qquad 1(2^2) + 2(2^3) + 3(2^4) + \cdots + (n-2)(2^{n-1}) + (n-1)(2^n) \\ \hline -S = & 2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^{n-1} - (n-1)(2^n) \end{array}$$

All of the positive terms on the right hand side form a geometric sequence of n - 1 terms with a first term of 2 and a common ratio of 2, which has a sum of $\frac{2(1-2^{n-1})}{1-2} = 2^n - 2$.

Simplifying, we have $S = -(2^n - 2) + (n-1)2^n = (n-1)2^n - 2^n + 2 = (n-2)2^n + 2$.

Finally, to get our final sum, we simply must add n to this to obtain

$$(n-2)2^n + n + 2$$

4) If the graphs of y = 2|x – a| + b and y = -2|x – c| + d intersect at both (5, 6) and (7, 8), what is the value of a – b + c – d?

Solution

The first graph looks like a 'V' while the second one looks like an upside down 'V', based on the slopes of 2 and -2, respectively. The "vertex" of the first V is (a, b) and the vertex of the second V is (c, d). Thus, the slope of the line segment connecting (a, b) and (7, 8) is 2 and the slope of the line segment connecting (5, 6) and (a, b) is -2. Similarly, we find that the slope between (5, 6) and (c, d) is 2 and the slope between (c, d) and (7, 8) is -2. This gives us four equations for these four slopes:

$$\frac{8-b}{7-a}=2, \qquad \frac{6-b}{5-a}=-2, \qquad \frac{6-d}{5-c}=2, \qquad \frac{8-d}{7-c}=-2$$

Solving the first system of two equations, we have:

$$8-b=2(7-a)=14-2a$$
$$6-b=-2(5-a)=-10+2a$$

By solving the first equation for b we find, $b=2a-6$. By substituting for b in the second equation, we get $6-(2a-6)=-10+2a$, thus $4a=22$ and $a=\frac{11}{2}$, $b=5$.

Solving the second system of two equations, we have:

$$6-d=2(5-c)=10-2c$$
$$8-d=-2(7-c)=-14+2c$$

Similarly, using the first equation, we find, $d=2c-4$. Substituting into the second equation we find $8-(2c-4)=-14+2c$, thus $4c=26$ and $c=\frac{13}{2}$, $d=9$.

It follows that the quantity $a-b+c-d=-2$.

Note: an easier solution exists utilizing the fact that the two Vs that meet form a parallelogram with parallel sides with slopes 2 and -2. We simply note that if we travel from (5, 6) to (a, b) and travel from (7, 8) to (c, d), due to the vectors of travel being exact opposites, the sum of the two vectors is 0i + 0j. Let the vector from (5, 6) to (a, b) be v_1, then we can express the point (a, b) as 5i + 6j + v_1. It follows that the point (c, d) is 7i + 8j - v_1. If we sum up the coordinates of the points (a, b) and (c, d) using these representations, we get the sum as

5i + 6j + v_1 + 7i + 8j - v_1 = 12i + 14j.

This sum is also (a+c)i + (b+d)j. Equating both components of these two representations, we find that a+c = 12 and b+d = 14, thus, $a-b+c-d=(a+c)-(b+d)=12-14=-2$.

5) During a particular tennis tournament, Sarah won three matches and lost none. These three matches increased her winning percentage by precisely 2. Determine the number of matches Sarah had won prior to the tournament, assuming that she had previously won at least one match. Is it possible to determine the total number of matches she had played prior to the tournament?

<u>Solution</u>

Let the number of matches Sarah won prior to the tournament be *W* and the number of matches she played in total before the tournament be *T*. Using the given information, we have the following equation:

$$\frac{W}{T} + .02 = \frac{W+3}{T+3}$$

Multiplying through by $50T(T+3)$ to clear all fractions, we find:

$$50(T+3)W + T(T+3) = 50(W+3)T$$
$$50TW + 150W + T^2 + 3T = 50TW + 150T$$
$$T^2 - 147T + 150W = 0$$

This is a quadratic equation in *T*, with a sum of roots of 147 and a product of roots that is a multiple of 150. Let T_1 be one of the two roots of this equation. Then we have that $T_1(147 - T_1)$ has 150 as a factor. There are 12 factors of 150: 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75 and 150. Without loss of generality, we have six cases to check:

1) $1 \mid T_1$ and $150 \mid (147 - T_1)$
2) $2 \mid T_1$ and $75 \mid (147 - T_1)$
3) $3 \mid T_1$ and $50 \mid (147 - T_1)$
4) $5 \mid T_1$ and $30 \mid (147 - T_1)$
5) $6 \mid T_1$ and $25 \mid (147 - T_1)$
6) $10 \mid T_1$ and $15 \mid (147 - T_1)$

Case #1 is out because its second constraint would force T_1 to be negative or make W = 0.

Case #2 has the solutions $T_1 = 72$ or $T_1 = 75$.

Case #3 has no solutions since neither 47 nor 97 is divisible by 3.

Case #4 doesn't have any solutions since two numbers both divisible by 5 can't add up to 147.

Case #5 gives the same two solutions as Case #2. (In attempting to satisfy the second equation, we try $T_1 = 22, 47, 72, 97$ and 122, of which only 72 is divisible by 6.)

Case #6, like Case #4 yields no solutions since two numbers that sum to 147 can't both be divisible by 5.

Thus, the number of total games played before the tournament was either 72 or 75. Regardless of which of these two totals was correct, the corresponding number of wins must satisfy the equation $150W = 72 \times 75$, thus $W = \frac{72\times 75}{150} = 36$.

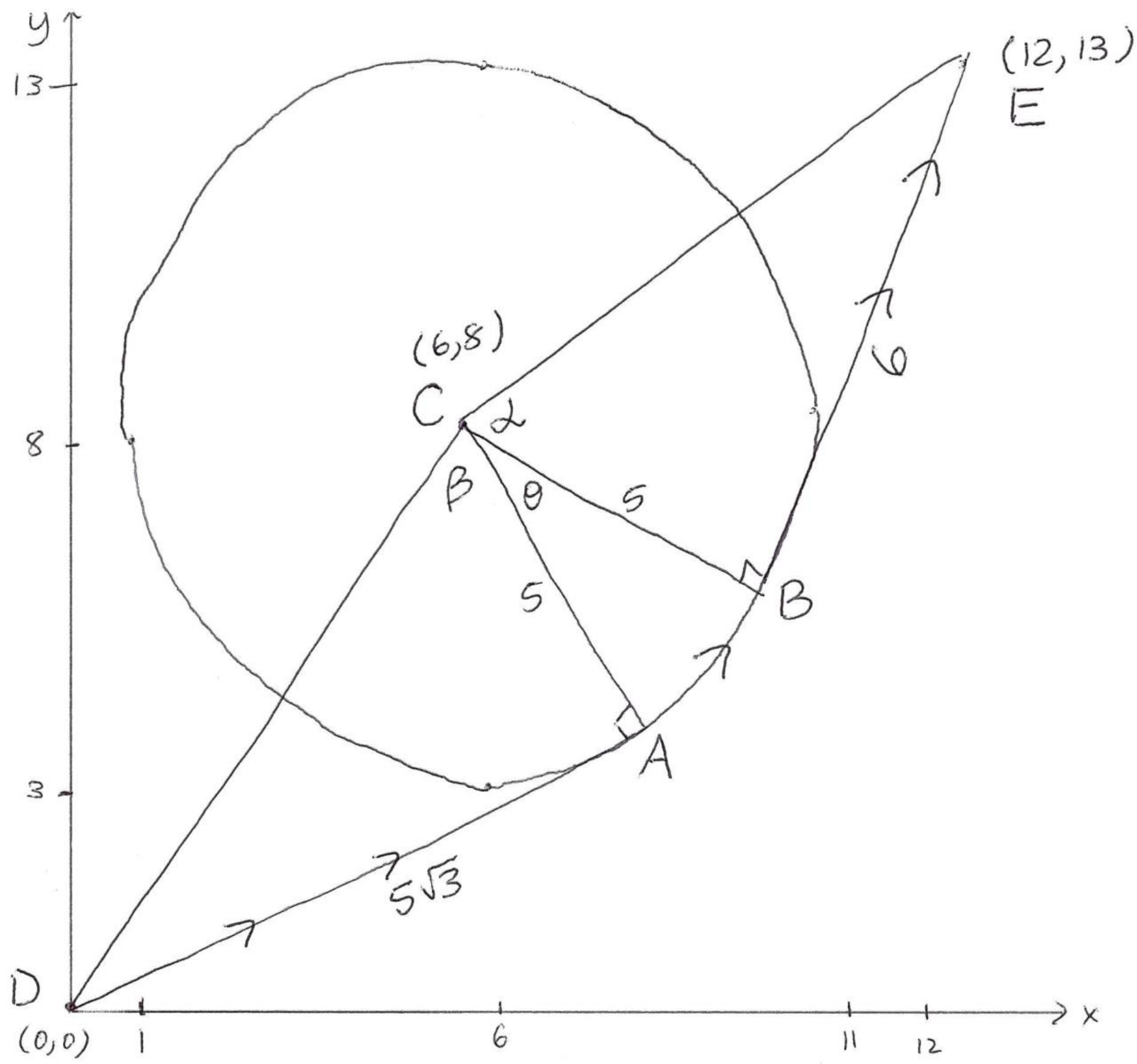
Sarah won exactly 36 matches prior to the tournament, but the total number of games she had played can not be uniquely determined. (But we do know it was either 72 or 75.)

Investigations

1) In the xy-plane, what is the length of the shortest path from (0, 0) to (12, 13) that does not go inside the circle $(x-6)^2 + (y-8)^2 = 25$?

Solution

The solution will reference the drawing shown below, with the shortest path drawn in. (Apologies for the inaccuracies; it was hand drawn.)



Since the circle is in the straight line path between the points, our shortest route will be to walk to the circle on a tangent line, trace a portion of the circle and then walk on another tangent line to the finish point. The triangle inequality can be used to prove that this route is the shortest. If we never touch the circle, our path can be seen as a path that goes around a larger circle, which is the result of "stretching out" the current path. (Think of a rubber band being tied from point D to point E around the circle centered at C.) If we do touch the circle, our first point of tangency, coming from D must be A or to the left of A. If it is the latter, we are required to walk around the circle for a greater distance than the path shown and will eventually reach A. In this case, the straight line distance to A is shorter than hitting the circle at another point and tracing the circle to A, since all straight line distances are shorter than any alternative. Any alternative to tracing the circle leaves the circle and hits the circle. As we use shorter and shorter segments to do this, their length approaches the length of the corresponding arc of the circle from above.

Since CAD and CBE are right triangles with the right angles at A and B, respectively, we can quickly use the Pythagorean Theorem in conjuction with the distance formula to fill in the side lengths AD and BE as $5\sqrt{3}$ and 6, respectively. What remains is determining the radian measure for angle θ. We can use the dot product between the vectors CD and CE to determine the angle measure of α + β + θ:

CD = -6i - 8j
CE = 6i + 5j

$$CD°CE = (-6i - 8k)°(6i + 5j) = -6(6) - 8(5) = -76 = |CD||CE|\cos(\alpha + \beta + \theta)$$

$|CD| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$, $|CE| = \sqrt{6^2 + 5^2} = \sqrt{61}$

Thus, $\cos(\alpha + \beta + \theta) = \frac{-76}{10\sqrt{61}}$.

Consequently, we can find $\sin(\alpha + \beta + \theta)$ and $\tan(\alpha + \beta + \theta)$:

$$\sin(\alpha + \beta + \theta) = \sqrt{1 - cos^2(\alpha + \beta + \theta)} = \sqrt{1 - \frac{76^2}{6100}} = \sqrt{\frac{6100 - 5776}{6100}} = \sqrt{\frac{324}{6100}} = \frac{9}{5\sqrt{61}}$$

$$\tan(\alpha + \beta + \theta) = \frac{\sin(\alpha + \beta + \theta)}{\cos(\alpha + \beta + \theta)} = \frac{\frac{18}{10\sqrt{61}}}{\frac{-76}{10\sqrt{61}}} = -\frac{9}{38}$$

Since $tan\alpha = \frac{6}{5}$, we have:

$$\tan(\beta + \theta) = \tan\big((\alpha + \beta + \theta) - \alpha\big) = \frac{\tan(\alpha + \beta + \theta) - tan\alpha}{1 + \tan(\alpha + \beta + \theta)\, tan\alpha}$$

Thus, $\tan(\beta + \theta) = \frac{-\frac{9}{38} - \frac{6}{5}}{1 - \frac{9}{38} \times \frac{6}{5}} = \frac{\frac{-273}{190}}{\frac{136}{190}} = \frac{-273}{136}$.

Finally, since $tan\beta = \sqrt{3}$, we have:

$$\tan(\theta) = \tan\big((\beta+\theta)-\beta\big) = \frac{\tan(\beta+\theta) - tan\beta}{1+\tan(\beta+\theta)\,tan\beta} = \frac{\frac{-273}{136} - \sqrt{3}}{1 - \frac{273\sqrt{3}}{136}} = \frac{273+136\sqrt{3}}{273\sqrt{3}-136}$$

It follows that one expression for the shortest distance between the points given that doesn't go into the circle given is $5\sqrt{3} + 6 + 5tan^{-1}(\frac{273+136\sqrt{3}}{273\sqrt{3}-136})$. This is approximately 19.5891546.

2) Let m and n be positive integers with m > n. Prove that $m^2 - n^2$, 2mn and $m^2 + n^2$ form a Pythagorean Triple. Determine the least set of further constraints on m and n that guarantees that the Pythagorean Triple designated is a primitive Pythagorean Triple. Note: a primitive Pythagorean Triple is one where the greatest common divisor of the three side lengths is 1.

Solution

First, we must show that the first two given items squared equals the third item squared. Since m and n are integers, it follows that the given values are integers also, so showing that the values satisfy the Pythagorean Theorem will show that they are a Pythagorean Triple.

$$(m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2$$

Let a = gcd(m, n) and let m = am' and n = an'. If a > 1, then we notice that the Pythagorean Triple produced by the three expressions all share a^2 as a common factor. Formally, we have:

$$m^2 - n^2 = (am')^2 - (an')^2 = a^2({m'}^2 - {n'}^2)$$
$$2mn = 2(am')(an') = a^2(2m'n')$$
$$m^2 + n^2 = (am')^2 + (an')^2 = a^2({m'}^2 + {n'}^2)$$

Thus, we see that if m and n share a common factor, the corresponding triple produced shares that factor squared. In order for the triple to be primitive, we see that gcd(m, n) = 1 is a requirement.

Unfortunately, the proof above doesn't show that if gcd(m, n) = 1, that the triple is primitive. It just shows that if the gcd isn't one, then the triple ISN'T a primitive one.

If we quickly look to make a parity argument, we see that if m and n are both odd, all three values will end up even, also producing a non-primitive triple. Let's prove this formally. Let m = 2a+1 and n = 2b+1 for arbitrary positive integers a and b. (Note that no Pythagorean triple has 1 in it, so we know that a and b are positive.)

$$m^2 - n^2 = (2a + 1)^2 - (2b + 1)^2 = 4a^2 + 4a + 1 - 4b^2 - 4b - 1 = 4(a^2 + b^2 - a - b)$$
$$2mn = 2(2a + 1)(2b + 1)$$
$$m^2 + n^2 = (2a + 1)^2 + (2b + 1)^2 = 4a^2 + 4a + 1 + 4b^2 + 4b + 1 = 4(a^2 + b^2 + a + b) + 2$$

Notice that all three are even. (We can factor 2 out from both terms in the last expression.)

Thus, we know that if a triple is to be primitive, we must have gcd(m, n) = 1, with exactly one of the two values being odd. It remains to be shown that no further restrictions are needed; namely, if these two items are true, that the corresponding triple IS primitive.

3) Simplify the expression

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\cdots+\sqrt{1+\frac{1}{2013^2}+\frac{1}{2014^2}}$$

Solution

The goal here is to obtain some sort of "telescopic" sum, or to discover a pattern to prove via mathematical induction. Let T_n be the n^{th} term in the sum. Thus, $T_1 = \frac{3}{2}$, $T_2 = \frac{7}{6}$, and $T_3 = \frac{13}{12}$. We see that each of these terms is rational and the denominator of the n^{th} term is simply n(n+1). The latter makes sense since we create a common denominator of n^2 and $(n+1)^2$ in the n^{th} term before taking the square root. Let's algebraically work out a simplified form for T_n:

$$T_n = \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} = \sqrt{\frac{n^2(n+1)^2+(n+1)^2+n^2}{n^2(n+1)^2}}$$
$$= \frac{\sqrt{n^4+2n^3+n^2+n^2+2n+1+n^2}}{n(n+1)}$$
$$= \frac{\sqrt{n^4+2n^3+3n^2+2n+1}}{n(n+1)}$$
$$= \frac{\sqrt{(n^2+n+1)^2}}{n(n+1)}$$
$$= \frac{n^2+n+1}{n(n+1)}$$
$$= \frac{n(n+1)+1}{n(n+1)}$$
$$= 1+\frac{1}{n(n+1)}$$
$$= 1+\frac{1}{n}-\frac{1}{n+1}$$

Note that we have two "creative" steps here. One is recognizing that $n^4 + 2n^3 + 3n^2 + 2n + 1$ is a perfect square that isn't commonly taught and the other is the partial fraction decomposition at the very end of the problem that will make evaluating our sum easier.

Now that we've worked out a simplified expression for each term, we can rewrite our summation and solve it as follows:

$$\sum_{k=1}^{2013}(1+\frac{1}{k}-\frac{1}{k+1}) =$$
$$\sum_{k=1}^{2013}1+\sum_{k=1}^{2013}\frac{1}{k}-\sum_{k=1}^{2013}\frac{1}{k+1} =$$

$$2013 + 1 + (\sum_{k=2}^{2013} \frac{1}{k}) - (\sum_{k=2}^{2014} \frac{1}{k}) =$$
$$2014 + \left(\sum_{k=2}^{2013} \frac{1}{k}\right) - \left(\sum_{k=2}^{2013} \frac{1}{k}\right) - \frac{1}{2014} =$$
$$2014 - \frac{1}{2014}$$

4) What is the smallest non-negative integer of the form

$$\pm 1^3 \pm 2^3 \pm 3^3 \pm \cdots \pm 2014^3,$$

for some choice of signs? Provide proof of this minimum as well as one choice of signs that satisfies it.

Solution

Note that the even terms don't affect the parity of the expression, only the odd terms do. There are exactly 1007 odd terms, thus the total sum of terms must be odd. This excludes 0 as a possible value of the expression.

We will now construct one possible choice of signs to achieve a sum of 1. There are many that are possible.

Consider any sequence of four terms in a row from the larger sequence. If our goal is to "minimize" the absolute value of the sum of these terms, it makes sense to make two terms positive and two negative. Furthermore, it probably makes sense to choose the same sign for the smallest and largest terms, to get as close to 0 as possible. Let's see what value we get with this choice of terms for an arbitrary starting integer a:

$$(a+3)^3 - (a+2)^3 - (a+1)^3 + a^3 = 9a^2 + 27a + 27 - 6a^2 - 12a - 8 - 3a^2 - 3a - 1$$
$$= 12a + 18$$

Now, let's take the next four terms, but flip all four signs to get a negative value, in an attempt to offset this positive value:

$$-(a+7)^3 + (a+6)^3 + (a+5)^3 - (a+4)^3 =$$
$$-((a+2)+7)^3 + ((a+4)+2)^3 + ((a+4)+1)^3 - (a+4)^3 =$$

$$= -(12(a+4)+18) = -12a - 30$$

Thus, adding eight consecutive terms together with these two choices of signs yields a value of

$$12a + 18 - (12a - 30) = -48$$

It follows that if we choose opposite signs for the next 8 consecutive integers, their sum would be 48. Thus, putting it all together, we find that for any 16 consecutive integers a through a + 15,

$$a^3 - (a+1)^3 - (a+2)^3 + (a+3)^3 - (a+4)^3 + (a+5)^3 + (a+6)^2 - (a+7)^3$$
$$-(a+8)^3 + (a+9)^3 + (a+10)^3 - (a+11)^3 + (a+12)^3 - (a+13)^3 - (a+14)^3 + (a+15)^3$$
$$= 0$$

Applying this principle to our given problem, we can use these choice of signs for every consecutive set of 16 integers from 15 to 2014, (There are $\frac{2000}{16} = 125$ of these.), obtaining a sum of 0 for this assignment of signs.

Now, our goal will be to get a value of 1 in choosing the remaining 14 signs. Given our previous work, we already know that:

$$-7^3 + 8^3 + 9^3 - 10^3 + 11^3 - 12^3 - 13^3 + 14^3 = 48$$

Now, we can use brute force to try to set the remaining signs. We first set 6's sign to be negative, and work from there:

$$-3^3 + 4^3 + 5^3 - 6^3 = -54$$

Putting this together, we get a sum of -6. Finally, we can offset this negative with $-1^3 + 2^3$.

Thus, we have proven that if we set our first 14 signs to $-,+,-,+,+,-,-,+,+,-,+,-,-,+$, and then set every remaining set of 16 signs to $+,-,-,+,-,+,+,-,-,+,+,-,+,-,-,+$, the corresponding sum will be 1, the smallest possible non-negative sum.

5) (a) Suppose that a power of 2 contains the substring 2014. What is the fewest possible number of digits after the '2014'?

(b) Is it possible for a power of 2 to begin with the four digits 2014?

Solution - Part (a)

Let's consider the possibilities from the smallest value (0) on up.

Any power of 2 greater than 2 itself is equivalent to 0 mod 4, but any number that ends in 2014 is equivalent to 2 mod 4. Thus, no power of two ends in 2014. More formally, any positive integer that ends in 2014 can be expressed as

10000n + 2014, where n is a integer.

10000n + 2014 ≡ 0 + 2 ≡ 2 (mod 4), since 10000 is divisible by 4.

Now, consider an integer that ends in 2014x, where x is a single digit.

We know that any power of 2 greater than 16 is equivalent to 0 mod 32. Let's consider any integer ending in 2014x mod 32. Any integer ending in 2014x can be expressed as

$100000n + 20140 + x$, with $0 \leq x < 10$.

$100000n + 20140 + x \equiv 0 + 12 + x \equiv (12 + x) \bmod 32$

The smallest positive value for x that makes this quantity equivalent to 0 mod 32 is 20. But, this is impossible since x is a digit. It follows that no digit can replace x to create a value divisible by 32, thus, no power of two ends in 2014x, for any digit x.

Attempting to create a similar argument for integers that end in 2014xy, where x and y are digits fails. The reason such an argument fails is that we would be forced to look at the integer mod 64, the highest power of 2 that evenly divides into 10^6. In this case, we have enough flexibility with both x and y that the modulus equation can be satisfied; there's no way to show that no solution exists.

To finish the proof, we ought to use brute force to find the first power of 2 that ends in 2014xy for some digits x and y. Since we only care about the remainder when dividing by 1000000, we can write a simple C program to do our search, taking the modulus of each intermediate value with 1000000.

What the modulus argument does tell us is that since $201400 \equiv 56 \pmod{64}$, it follows that if a power of two ends in 2014xy, it must end in either 201408 or 201472, corresponding to the two positive integers less than 100 that we can add to 201400 to create an integer divisible by 64.

After running the C program shown below

```
#include <stdio.h>

#define LOW 201400
#define HIGH 201499
#define MAX 10000

int main() {

    int i, lastDigits = 1;

    for (i=0; i<MAX; i++) {
        if (lastDigits >= LOW && lastDigits <= HIGH)
            printf("power %d, ending = %d\n", i, lastDigits);
        lastDigits = (lastDigits*2)%1000000;
    }

    return 0;
}
```

we find that 2^{437} ends in 201472 and 2^{3983} ends in 201408.

Since the behavior of successive powers of 2 mod 10^6 is cyclic, we find a period of 12,500, implying that exactly one of every eight possible combinations of can be produced as the last six digits of a perfect power of two.

<u>Solution - Part (b)</u>
This turns out to be true due to the irrationality of $\log_{10}2$. We will provide a proof by contradiction to show prove the assertion. Though specific values will be plugged into this proof, we can slightly modify the proof to apply to any set of starting digits one could choose.

Consider any number that starts with 2014. It can be written in scientific notation as $x(10^n)$, where $2.014 \leq x < 2.015$ and n is a positive integer. Consider taking the log of this quantity base 10:

$$log_{10}\left(x(10^n)\right) = log_{10}x + log_{10}10^n = n + log_{10}x$$

The fractional part of this expression, denoted as $\{n + log_{10}x\} = log_{10}x$. Our goal is to prove that there exists some integer m for which $log_{10}2.014 \leq \{log_{10}2^m\} < log_{10}2.015$.

Let $\frac{1}{c} = log_{10}2.015 - log_{10}2.014$. We will now prove that in any interval of size c (or greater), there must be at least one term of the form $\{log_{10}2^m\}$, for some positive integer m.

Let us consider the $\lceil c+1 \rceil$ terms $\{log_{10}2^1\}$, $\{log_{10}2^2\}$, ..., $\left\{log_{10}2^{\lceil c+1 \rceil}\right\}$ in terms of where they reside in the $\lceil c \rceil$ intervals $[0, \frac{1}{\lceil c \rceil})$, $[\frac{1}{\lceil c \rceil}, \frac{2}{\lceil c \rceil})$, ..., $[\frac{\lceil c-1 \rceil}{\lceil c \rceil} 1)$. By the Pigeonhole Principle, one of these intervals must contain at least two of the given terms. Without loss of generality, let these terms be $\left\{log_{10}2^i\right\}$ and $\left\{log_{10}2^j\right\}$, with the previous term being strictly smaller than the latter term.

Note that the two terms can't be equal as that contradicts the fact that $\log_{10}2$ is irrational. To see this, note that if these terms are equal, then dividing the corresponding powers of 2 will yield a perfect power of 10, giving an equation of the form $2^{i-j} = 10^k$, where i, j and k are integers. This implies that k = (i - j)$\log_{10}2$, implying that $log_{10}2 = \frac{k}{i-j}$, a rational number, which we know isn't possible.

Let $c' = \left\{log_{10}2^j\right\} - \left\{log_{10}2^i\right\}$. We know that $c' < \frac{1}{c}$, since both terms are contained in an interval of size $\frac{1}{\lceil c \rceil}$. Now, let's consider two cases:

Case 1: j > i. In this case, we have $c' = \{log_{10}2^{j-i}\}$. Let $s = j - i$. This means that for some integer k, if $\{log_{10}2^k\} = x$, then$\{log_{10}2^{k+s}\} = \{x + c'\}$. Since $\log_{10}2^0 = 0$, we find that as we repeatedly use this exponent s, that we have $\{log_{10}2^{as}\} = ac'$, for all integers a, $0 \leq a < \frac{1}{c'}$. Thus, each "multiple" of c' appears as a fractional part of some exponent. Formally, let a be the largest integer in the given range such that $\{log_{10}2^{as}\} < log_{10}2.014$. By definition,

$\{log_{10}2^{(a+1)s}\} > log_{10}2.014$. But, $\{log_{10}2^{(a+1)s}\} = \{log_{10}2^{as}\} + c' < log_{10}2.014 + c' < log_{10}2.014 + \frac{1}{c} = log_{10}2.015$, proving that there exists a power of 2 for which the fractional part of its base 10 logarithm resides in the range $[log_{10}2.014, log_{10}2.015)$, implying that some power of 2 starts with 2014 in this case.

Case 2: j < i. In this case we have $1 - c' = \{log_{10}2^{i-j}\}$. Let s = i - j. This means that if $\{log_{10}2^{k}\} = x$, then$\{log_{10}2^{k+s}\} = \{x - c'\}$. Since $\log_{10}2^0 = 0$, we find that as we repeatedly use this exponent s, that we have $\{log_{10}2^{as}\} = 1 - ac'$, for all integers a, $0 \leq a < \frac{1}{c'}$. Thus, counting "backwards" from 1, each multiple of c' appears as a fraction part of some exponent. Formally, let a be the smallest integer in the given range such that $\{log_{10}2^{as}\} \geq log_{10}2.015$. By definition, $\{log_{10}2^{(a+1)s}\} < log_{10}2.015$. But, $\{log_{10}2^{(a+1)s}\} = \{log_{10}2^{as}\} - c' \geq log_{10}2.015 - c' > log_{10}2.015 - \frac{1}{c} = log_{10}2.014$, proving that there exists a power of 2 for which the fractional part of its base 10 logarithm resides in the range $[log_{10}2.014, log_{10}2.015)$, implying that some power of 2 starts with 2014 in this case as well.

Since we've shown the existence of a power of 2 such that the fractional part of its base 10 logarithm lies in the range, $[log_{10}2.014, log_{10}2.015)$, it follows that there must exist a power of 2 that starts with the digits 2014.

The following python program prints out the first power of 2 that starts with 2014:

```
def start(number, mystr):

    numberstr = str(number)
    if numberstr[:len(mystr)] == mystr:
        return True
    return False

def main():

    ans = 1
    i = 0
    while not(start(ans,"2014")):
        ans = ans*2
        i += 1
    print(i,ans)

main()
```

Note: Python was used here because it uses built in large integers, so no special API or syntax is necessary to carry out multiplication of large integers.

Running the program we find that the first power of 2 that starts with 2014 is a 2340 digit number. Namely,

2^{7771} =

201417084271274463294263673947471555051170526061704116586733068266977697508849
445982646772639189657467401428453651677475855997072327266421050186626004910624
299346008579269970081537138257476219109648584491845180551777448704053270311528
686680394769735852034655896048885642669208407836065849685451912792474159000283
470540380689006470484748260061175972111947524048446056794917669556363019835768
430276481457131118151060189669706161436074055641335915175682674211913706070505
840748558161040724278013801824767994302386579058305907842178265363226602808083
013455962481347233564666978474210311303223415688744686479472172075965692620410
194588946864725225307614147347533144615863916519999056393567070494490845529000
579810231299951528463043508087307172003417748469511117243033752814276956948343
274205030518373675741972753601206341421981385645959017977723266310805643726449
782559355674230789471359480076743404900276217900004019384301643741589488532939
065326205724401070344304779177720607175668310709774264483174829634400254488642
742959799896289796804074606984674302131502782007859708583191326019755089677128
393345569719876624786794496504243645447458005583404013270251839001618506877689
810392402527196474683002307220248623850918705666378166350980069950241985185038
005582492488464498764819593186349760995846311836328730657765984465475786075557
138549762932391395494903271147312009619481455286874424348362612678745445516283
081911653560881631859067902543828801470861327831423084118299511239230720644982
603742829328488449768993620499819974642569924720161630328891872396737124661616
538584520936589133529167220650395206257445931399736626283158201921935277354509
774493077044525847092133324844683822084288650231356041650429117496980344217929
758966217910685704183037466133544919693097561440065591919712363848701531098777
039126818307089510723886840409930390847122411517914119577848708839272591159443
062884359169410699100379999195646789170852183083804354606903746455274194768361
200882503563242889985446963585856225435139082316386704547645335051256604516881
467759289776811181098790546355519137457942669349966414206259255454191801889624
659655346777449337333969511512720690216877286944586323326849930576732783390911
235248316289760052323579467396931362561816959057374349772862725226289531523909
087402321363006287202033055534051773343845742939109282585563370693225347022848