Chaotic Iteration for the Depends On Analysis

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Abstract

An example of computing the "Depends On" analysis discussed in class.

1 The Depends On Analysis

1.1 Definitions

We say that a variable y *may have influenced the value of a variable* x if changing the value of y might change the value of x in some execution. In this case we say that x *may depend on* y.
(This dependency information could, for example, be used to help an optimizing compiler decide if two statements could be reordered or run in parallel.)
As discussed in class, the "Depends On" (DO) analysis concerns the question of what variables a variable may depend on at each program point.

1.1.1 Example Program

In class we discussed the following example program to illustrate the analysis.

```
[i := n]^1;
[j := i]^2;
if [i > 0]^3 then [m := j+1]^4 else [m := j-1]^5;
[j := m-i]^6;
[k := j]^7
```

In this example, at the exit to block 1, `i` depends on `n`. At the exit from block 2, `j` depends on both `i` and `n`, and `i` still depends on `n`. At exit from block 3, the same dependencies hold as on exit from block 2. At exit from block 4, in addition to the dependencies that hold on exit from block 3, `m` depends on `j`, `i`, and `n`, and the same holds on exit from block 5. At exit from block 6, all the dependencies that hold on exits from blocks 4 and 5 hold, except that `j` depends on `m`, `i`, `j`, and `n`. At exit from block 7, all the dependencies that hold on exits from block 6, and in addition, `k` depends on `j`, `m`, `i`, and `n`.

1.2 Mathematical Treatment

The analysis is a forward "may" analysis.

1.2.1 Property Space

We use as a property space (finite) binary relations between variable names (although one could also use an isomorphic property space, such as finite mappings from variable names to sets of variable names). That is, we use as the property space:

$$(L, \bigsqcup) = (\mathcal{P}(\mathbf{Var}_\star \times \mathbf{Var}_\star), \bigcup).$$

Since L is a finite set, the ordering induced by $\bigcup$ is a complete partial order. That is, this property space has the ascending chain property.

1.2.2 Dataflow Equations

We can define the DO analysis as a set of dataflow equations in the usual way by first defining the $\mathit{kill}_{\mathsf{DO}}$ and $\mathit{gen}_{\mathsf{DO}}$ functions. The $\mathit{gen}_{\mathsf{DO}}$ function is explicitly passed the binary relation on variables that holds on entry to the elementary block; this makes the result of $\mathit{gen}_{\mathsf{DO}}$ explictly depend on the entry information passed (in argument R), avoiding an implicit (and obscure) dependence on the entry information for the block. Also note that the transitive closure of the relation passed to $\mathit{gen}_{\mathsf{DO}}$, notated R^+ is used in the definition of the $\mathit{gen}_{\mathsf{DO}}$ function for assignments.[1]

$\mathit{kill}_{\mathsf{DO}} : \mathbf{Blocks}_\star \to L$
$\mathit{kill}_{\mathsf{DO}}([x := a]^\ell) = \{(x,z) \mid z \in \mathbf{Var}_\star\}$
$\mathit{kill}_{\mathsf{DO}}([\mathtt{skip}]^\ell) = \emptyset$
$\mathit{kill}_{\mathsf{DO}}([b]^\ell) = \emptyset$

$\mathit{gen}_{\mathsf{DO}} : L \to \mathbf{Blocks}_\star \to L$
$\mathit{gen}_{\mathsf{DO}}(R)([x := a]^\ell) = \{(x,y) \mid y \in FV(a)\} \cup \{(x,z) \mid y \in FV(a), (y,z) \in R^+\}$
$\mathit{gen}_{\mathsf{DO}}(R)([\mathtt{skip}]^\ell) = \emptyset$
$\mathit{gen}_{\mathsf{DO}}(R)([b]^\ell) = \emptyset$

Using the above definitions, we can define the analysis schematically as follows. (Note that $s_1 \setminus s_2$ means the subtraction of s_2 from s_1, that is: $s_1 \setminus s_2 = \{w \mid w \in s_1, w \notin s_2\}$.)

$$\mathsf{DO}_{\mathit{entry}}(\ell) = \begin{cases} \emptyset, & \text{if } \ell = \mathit{init}(S_\star) \\ \bigcup\{\mathsf{DO}_{\mathit{exit}}(\ell') \mid (\ell',\ell) \in \mathit{flow}(S_\star)\}, & \text{otherwise} \end{cases}$$
$$\mathsf{DO}_{\mathit{exit}}(\ell) = (\mathsf{DO}_{\mathit{entry}}(\ell) \setminus \mathit{kill}_{\mathsf{DO}}(B^\ell)) \cup \mathit{gen}_{\mathsf{DO}}(\mathsf{DO}_{\mathit{entry}}(\ell))(B^\ell)$$
where $B^\ell \in \mathit{blocks}(S_\star)$

1.2.3 Functional Representing the Dataflow Equations for the Example Program

In the example program shown in Sec. 1.1.1, there are 7 elementary blocks, and since there are entry and exit points for each of these blocks, the functional encoding the dataflow equations will have 14 components. So we define for the example program $F(\vec{DO}) = (F_1(\vec{DO}), \ldots, F_{14}(\vec{DO}))$ where $\vec{DO}$ is a 14-tuple of elements of L. The components of F are defined in Figure 1.

1.3 Solution for the Example Program

In this section we use Chaotic Iteration to find the least solution (in the pointwise subset ordering on 14-tuples, $\sqsubseteq$). We want the least solution, because that is the most precise solution available (as is true for all "may" analyses).
So to use Chaotic iteration, let each DO_i be $\emptyset$ (for $i \in \{1,\ldots,14\}$). Possible steps of the Chaotic Iteration for this example follow, starting from the beginning of the example program (since this is a forward analysis), but noting that no changes can be made to DO_1, due to the definition of F_1.
To take a step with the exit of block 1, we use the following lemma.

Lemma 1 *Let the F_2 be as in Figure 1 and let each $DO_i = \emptyset$. Then $F_2(DO_1,\ldots,DO_{14}) = \{(\mathtt{i},\mathtt{n})\}$.*

Proof: Assume that the F_2 is as in Figure 1 and that the DO_i are all empty. We calculate as follows.

$F_2(DO_1,\ldots,DO_{14})$
$=$ ⟨by definition of F_2⟩
$(DO_1 \setminus (\{\mathtt{i}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{i},\mathtt{n})\}$
$=$ ⟨by assumption $DO_1 = \emptyset$⟩
$(\emptyset \setminus (\{\mathtt{i}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{i},\mathtt{n})\}$
$=$ ⟨by set theory⟩
$\{(\mathtt{i},\mathtt{n})\}$ ∎

[1] Using the transitive closure is a difference from what we did in class, but seems necessary to obtain the right results.

entry 1:
$$F_1(DO_1, \ldots, DO_{14}) = \emptyset$$

exit 1:
$$\begin{aligned} F_2(DO_1, \ldots, DO_{14}) &= (DO_1 \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{i} := \mathtt{n}]^1) \cup \mathit{gen}_{\mathsf{DO}}(DO_1)([\mathtt{i} := \mathtt{n}]^1) \\ &= (DO_1 \setminus (\{\mathtt{i}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{i}, \mathtt{n})\} \end{aligned}$$

entry 2:
$$F_3(DO_1, \ldots, DO_{14}) = DO_2$$

exit 2:
$$\begin{aligned} F_4(DO_1, \ldots, DO_{14}) &= (DO_3 \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{j} := \mathtt{i}]^2) \cup \mathit{gen}_{\mathsf{DO}}(DO_3)([\mathtt{j} := \mathtt{i}]^2) \\ &= (DO_3 \setminus (\{\mathtt{j}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, z) \mid (\mathtt{i}, z) \in DO_3^+\} \end{aligned}$$

entry 3:
$$F_5(DO_1, \ldots, DO_{14}) = DO_4$$

exit 3:
$$\begin{aligned} F_6(DO_1, \ldots, DO_{14}) &= (DO_5 \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{i\ >\ 0}]^3) \cup \mathit{gen}_{\mathsf{DO}}(DO_5)([\mathtt{i\ >\ 0}]^3) \\ &= DO_5 \end{aligned}$$

entry 4:
$$F_7(DO_1, \ldots, DO_{14}) = DO_6$$

exit 4:
$$\begin{aligned} F_8(DO_1, \ldots, DO_{14}) &= (DO_7 \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{m} := \mathtt{j+1}]^4) \cup \mathit{gen}_{\mathsf{DO}}(DO_7)([\mathtt{m} := \mathtt{j+1}]^4) \\ &= (DO_7 \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m}, \mathtt{j})\} \cup \{(\mathtt{m}, z) \mid (\mathtt{j}, z) \in DO_7^+\} \end{aligned}$$

entry 5:
$$F_9(DO_1, \ldots, DO_{14}) = DO_6$$

exit 5:
$$\begin{aligned} F_{10}(DO_1, \ldots, DO_{14}) &= (DO_9 \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{m} := \mathtt{j-1}]^5) \cup \mathit{gen}_{\mathsf{DO}}(DO_9)([\mathtt{m} := \mathtt{j-1}]^5) \\ &= (DO_9 \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m}, \mathtt{j})\} \cup \{(\mathtt{m}, z) \mid (\mathtt{j}, z) \in DO_9^+\} \end{aligned}$$

entry 6:
$$F_{11}(DO_1, \ldots, DO_{14}) = DO_8 \cup DO_{10}$$

exit 6:
$$\begin{aligned} F_{12}(DO_1, \ldots, DO_{14}) &= (DO_{11} \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{m} := \mathtt{j-i}]^6) \cup \mathit{gen}_{\mathsf{DO}}(DO_{11})([\mathtt{m} := \mathtt{j-i}]^6) \\ &= (DO_{11} \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i})\} \\ &\qquad \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j}, \mathtt{i}\}, (y, z) \in DO_{11}^+\} \end{aligned}$$

entry 7:
$$F_{13}(DO_1, \ldots, DO_{14}) = DO_{12}$$

exit 7:
$$\begin{aligned} F_{14}(DO_1, \ldots, DO_{14}) &= (DO_{13} \setminus \mathit{kill}_{\mathsf{DO}}([\mathtt{k} := \mathtt{j}]^7) \cup \mathit{gen}_{\mathsf{DO}}(DO_{13})([\mathtt{k} := \mathtt{j}]^7) \\ &= (DO_{13} \setminus (\{\mathtt{k}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m}, \mathtt{j})\} \cup \{(\mathtt{k}, z) \mid (\mathtt{j}, z) \in DO_{13}^+\} \end{aligned}$$

Figure 1: Formulation of the dataflow equations for the example program.

Thus we can take the following step.

$$
\begin{array}{ll}
 & DO_2 \\
:= & \quad \langle \text{by Lemma 1, } F_2(DO_1, \ldots, DO_{14}) \neq \emptyset = DO_2\text{, so can take this step} \rangle \\
 & \{(\mathtt{i}, \mathtt{n})\}
\end{array}
$$

The previous step allows a step at the entry of block 2.

Lemma 2 *Let F_3 be as in Figure 1 and let the DO_i be as above. Then $F_3(DO_1, \ldots, DO_{14}) = \{(\mathtt{i}, \mathtt{n})\}$.*

Proof:

$$
\begin{array}{ll}
 & F_3(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_3 \rangle \\
 & DO_2 \\
= & \quad \langle \text{by assumption the value of } DO_2 \text{ is } \{(\mathtt{i}, \mathtt{n})\}. \rangle \\
 & \{(\mathtt{i}, \mathtt{n})\} \quad \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_3 \\
:= & \quad \langle \text{by Lemma 2, } F_3(DO_1, \ldots, DO_{14}) = \{(\mathtt{i}, \mathtt{n})\}\text{, so can take this step} \rangle \\
 & \{(\mathtt{i}, \mathtt{n})\}
\end{array}
$$

The next step is justified by the following lemma.

Lemma 3 *Let F_4 be as in Figure 1 and let the DO_i be as above. Then $F_4(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}$.*

Proof: Assume that the F_4 is as in Figure 1 and that the DO_i are as above. We calculate as follows.

$$
\begin{array}{ll}
 & F_4(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_4 \rangle \\
 & (DO_3 \setminus (\{\mathtt{j}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, z) \mid (\mathtt{i}, z) \in DO_3\} \\
= & \quad \langle \text{by assumption } DO_3 = \{(\mathtt{i}, \mathtt{n})\} \rangle \\
 & (\{(\mathtt{i}, \mathtt{n})\} \setminus (\{\mathtt{j}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, z) \mid (\mathtt{i}, z) \in \{(\mathtt{i}, \mathtt{n})\}\} \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{i}, \mathtt{n})\} \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \quad \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_4 \\
:= & \quad \langle \text{by Lemma 3, } F_4(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \neq \emptyset = DO_4\text{, so can take this step} \rangle \\
 & \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}
\end{array}
$$

The previous step allows a step at the entry of block 3.

Lemma 4 *Let F_5 be as in Figure 1 and let the DO_i be as above. Then $F_5(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}$.*

Proof:

$$
\begin{array}{ll}
 & F_5(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_5 \rangle \\
 & DO_4 \\
= & \quad \langle \text{by assumption the value of } DO_4 \text{ is } \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}. \rangle \\
 & \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \quad \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_5 \\
:= & \quad \langle \text{by Lemma 3, } F_5(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \neq \emptyset = DO_5\text{, so can take this step} \rangle \\
 & \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}
\end{array}
$$

The previous step allows a step at the exit of block 3.

Lemma 5 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then $F_6(DO_1, \ldots, DO_{14}) = \{(j, i), (i, n)\}$.*

Proof:

$$
\begin{array}{ll}
 & F_6(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_6 \rangle \\
 & DO_5 \\
= & \quad \langle \text{by assumption the value of } DO_5 \text{ is } \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\}. \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_6 \\
:= & \quad \langle \text{by Lemma 5, } F_6(DO_1, \ldots, DO_{14}) = \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \neq \emptyset = DO_6\text{, so can take this step} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\}
\end{array}
$$

The previous step allows a step at the entry of block 4.

Lemma 6 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then $F_7(DO_1, \ldots, DO_{14}) = \{(j, i), (i, n)\}$.*

Proof:

$$
\begin{array}{ll}
 & F_7(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_7 \rangle \\
 & DO_6 \\
= & \quad \langle \text{by assumption the value of } DO_6 \text{ is } \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\}. \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_7 \\
:= & \quad \langle \text{by Lemma 7, } F_7(DO_1, \ldots, DO_{14}) = \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \neq \emptyset = DO_7\text{, so can take this step} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\}
\end{array}
$$

The next step, for the exit of block 4, is justified by the following lemma.

Lemma 7 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then $F_8(DO_1, \ldots, DO_{14}) = \{(j, i), (i, n), (m, j), (m, i), (m, n)\}$.*

Proof: Assume that the F_i are as in Figure 1 and that the DO_i are as above. We calculate as follows.

$$
\begin{array}{ll}
 & F_8(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_8 \rangle \\
 & (DO_7 \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m},\mathtt{j})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j},\mathtt{i}\}, (y, z) \in DO_7^+ \\
= & \quad \langle \text{by assumption } DO_7 = \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \rangle \\
 & (\{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j},\mathtt{i}\}, (y, z) \in (\{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\} \cup \{(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j},\mathtt{i}\}, (y, z) \in (\{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j},\mathtt{i}\}, (y, z) \in (\{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i})\} \cup \{(\mathtt{m},\mathtt{i}),(\mathtt{m},\mathtt{n})\} \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{m},\mathtt{n})\} \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_8 \\
:= & \quad \langle \text{by Lemma 7, } F_8(DO_1, \ldots, DO_{14}) = \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{m},\mathtt{n})\} \neq \emptyset = DO_8\text{, so can take this step} \rangle \\
 & \{(\mathtt{j},\mathtt{i}),(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{m},\mathtt{n})\}
\end{array}
$$

Similarly, a step for the entry of block 5 can be taken, which is justified by the following lemma.

Lemma 8 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then $F_9(DO_1, \ldots, DO_{14}) = \{(j, i), (i, n)\}$.*

Proof:

$$
\begin{array}{ll}
& F_9(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_9 \rangle \\
& DO_6 \\
= & \quad \langle \text{by assumption the value of } DO_6 \text{ is } \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}. \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
& DO_9 \\
:= & \quad \langle \text{by Lemma 8, } F_9(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}, \text{ so can take this step} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\}
\end{array}
$$

The previous step allows a step at the exit of block 5.

Lemma 9 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then* $F_{10}(DO_1, \ldots, DO_{14}) = \{(j, i), (i, n), (m, j), (m, i), (m, n)\}$.

Proof: Assume that the F_i are as in Figure 1 and that the DO_i are as above. We calculate as follows.

$$
\begin{array}{ll}
& F_{10}(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_{10} \rangle \\
& (DO_9 \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j}, \mathtt{i}\}, (y, z) \in DO_9^+ \\
= & \quad \langle \text{by assumption } DO_9 = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \rangle \\
& (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \setminus (\{\mathtt{m}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j}, \mathtt{i}\}, (y, z) \in (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\} \cup \{(\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j}, \mathtt{i}\}, (y, z) \in (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i})\} \cup \{(\mathtt{m}, z) \mid y \in \{\mathtt{j}, \mathtt{i}\}, (y, z) \in (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i})\} \cup \{(\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \\
= & \quad \langle \text{by set theory} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
& DO_{10} \\
:= & \quad \langle \text{by Lemma 9, } F_{10}(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\}, \text{ so can take this step} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\}
\end{array}
$$

The above steps allow the iteration to take a step at the entry to block 6.

Lemma 10 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then* $F_{11}(DO_1, \ldots, DO_{14}) = \{(j, i), (i, n), (m, j), (m, i), (m, n)\}$.

Proof:

$$
\begin{array}{ll}
& F_{11}(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_{11} \rangle \\
& DO_8 \cup DO_{10} \\
= & \quad \langle \text{by assumption the values of } DO_8 \text{ and } DO_{10} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \cup \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \\
= & \quad \langle \text{by set theory} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
& DO_{11} \\
:= & \quad \langle \text{by Lemma 10, } F_{11}(DO_1, \ldots, DO_{14}) = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\}, \text{ so can take this step} \rangle \\
& \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\}
\end{array}
$$

The previous step allows a step at the exit of block 6.

Lemma 11 *Let the F_i be as in Figure 1 and let the DO_i be as above. Then* $F_{12}(DO_1, \ldots, DO_{14}) = \{(i, n), (m, j), (m, i), (j, m), (j, i), (j, j), (j, n)\}$.

Proof: Assume that the F_i are as in Figure 1 and that the DO_i are as above. We calculate as follows.

$$
\begin{array}{ll}
 & F_{12}(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_{12} \rangle \\
 & (DO_{11} \setminus (\{\mathtt{j}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, z) \mid y \in \{\mathtt{m}, \mathtt{i}\}, (y, z) \in DO_{11}^+ \\
= & \quad \langle \text{by assumption } DO_{11} = \{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \rangle \\
 & \ (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \setminus (\{\mathtt{j}\} \times \mathbf{Var}_\star)) \\
 & \ \cup \{(\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, z) \mid y \in \{\mathtt{m}, \mathtt{i}\}, (y, z) \in (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
 & \ \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\} \\
 & \ \cup \{(\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, z) \mid y \in \{\mathtt{m}, \mathtt{i}\}, (y, z) \in (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
 & \ \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i})\} \\
 & \ \cup \{(\mathtt{j}, z) \mid y \in \{\mathtt{m}, \mathtt{i}\}, (y, z) \in (\{(\mathtt{j}, \mathtt{i}), (\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{m}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i})\} \cup \{(\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{n})\} \\
= & \quad \langle \text{by set theory} \rangle \\
 & \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\} \ \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_{12} \\
:= & \quad \langle \text{by Lemma 9, } F_{12}(DO_1, \ldots, DO_{14}) = \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\}\text{, so can take this step} \rangle \\
 & \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\}
\end{array}
$$

The previous step allows a step at the entry of block 7.

Lemma 12 *Let F_{13} be as in Figure 1 and let the DO_i be as above. Then* $F_{13}(DO_1, \ldots, DO_{14}) = \{(i, n), (m, j), (m, i), (j, m), (j, i), (j, j), (j, n)\}$.

Proof:

$$
\begin{array}{ll}
 & F_{13}(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_{13} \rangle \\
 & DO_{12} \\
= & \quad \langle \text{by assumption the value of } DO_{12} \text{ is } \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\}. \rangle \\
 & \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\} \ \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_{13} \\
:= & \quad \langle \text{by Lemma 12, } F_{13}(DO_1, \ldots, DO_{14}) = \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\}\text{, so can take this step} \rangle \\
 & \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\}
\end{array}
$$

The next step is justified by the following lemma.

Lemma 13 *Let F_{14} be as in Figure 1 and let the DO_i be as above. Then* $F_{14}(DO_1, \ldots, DO_{14}) = \{(i, n), (m, j), (m, i), (j, m), (j, i), (j, j), (j, n), (k, j), (k, m), (k, i), (k, n)\}$.

Proof: Assume that the F_{14} is as in Figure 1 and that the DO_i are as above. We calculate as follows.

$$
\begin{array}{ll}
 & F_{14}(DO_1, \ldots, DO_{14}) \\
= & \quad \langle \text{by definition of } F_{14} \rangle \\
 & (DO_{13} \setminus (\{\mathtt{k}\} \times \mathbf{Var}_\star)) \cup \{(\mathtt{k}, \mathtt{j})\} \cup \{(\mathtt{k}, z) \mid (\mathtt{j}, z) \in DO_{13}^+\} \\
= & \quad \langle \text{by assumption } DO_{13} = \{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\} \rangle \\
 & \ (\{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\} \setminus \setminus (\{\mathtt{k}\} \times \mathbf{Var}_\star)) \\
 & \ \cup \{(\mathtt{k}, \mathtt{j})\} \cup \{(\mathtt{k}, z) \mid (\mathtt{j}, z) \in (\{(\mathtt{i}, \mathtt{n}), (\mathtt{m}, \mathtt{j}), (\mathtt{m}, \mathtt{i}), (\mathtt{j}, \mathtt{m}), (\mathtt{j}, \mathtt{i}), (\mathtt{j}, \mathtt{j}), (\mathtt{j}, \mathtt{n})\})^+ \\
= & \quad \langle \text{by set theory} \rangle
\end{array}
$$

$$
\begin{array}{ll}
 & \{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n})\} \cup \{(\mathtt{k},\mathtt{j})\} \\
 & \cup \{(\mathtt{k},z) \mid (\mathtt{j},z) \in (\{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n})\})^+ \\
= & \langle \text{by set theory} \rangle \\
 & \{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n}),(\mathtt{k},\mathtt{j})\} \\
 & \cup \{(\mathtt{k},z) \mid (\mathtt{j},z) \in (\{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n})\})^+ \\
= & \langle \text{by set theory} \rangle \\
 & \{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n}),(\mathtt{k},\mathtt{j})\} \\
 & \cup \{(\mathtt{k},\mathtt{m}),(\mathtt{k},\mathtt{i}),(\mathtt{k},\mathtt{j}),(\mathtt{k},\mathtt{n})\} \\
= & \langle \text{by set theory} \rangle \\
 & \{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n}),(\mathtt{k},\mathtt{j}),(\mathtt{k},\mathtt{m}),(\mathtt{k},\mathtt{i}),(\mathtt{k},\mathtt{n})\} \; \blacksquare
\end{array}
$$

$$
\begin{array}{ll}
 & DO_{14} \\
:= & \langle \text{by Lemma 13, } F_{14}(DO_1,\ldots,DO_{14}) \neq \emptyset = DO_{14}\text{, so can take this step} \rangle \\
 & \{(\mathtt{i},\mathtt{n}),(\mathtt{m},\mathtt{j}),(\mathtt{m},\mathtt{i}),(\mathtt{j},\mathtt{m}),(\mathtt{j},\mathtt{i}),(\mathtt{j},\mathtt{j}),(\mathtt{j},\mathtt{n}),(\mathtt{k},\mathtt{j}),(\mathtt{k},\mathtt{m}),(\mathtt{k},\mathtt{i}),(\mathtt{k},\mathtt{n})\}
\end{array}
$$

After these steps, no more steps are possible, because the example program has no loops. So the values of the DO_i constitute the least solution to the equations.