

CHAPTER 2

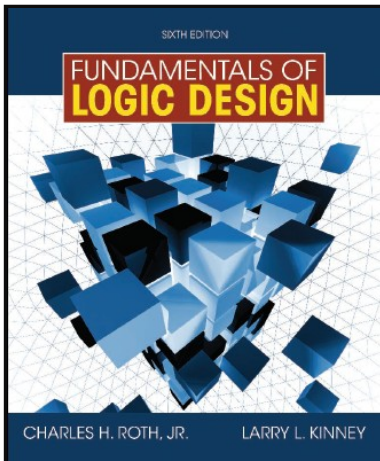
BOOLEAN ALGEBRA

This chapter in the book includes:

Objectives

Study Guide

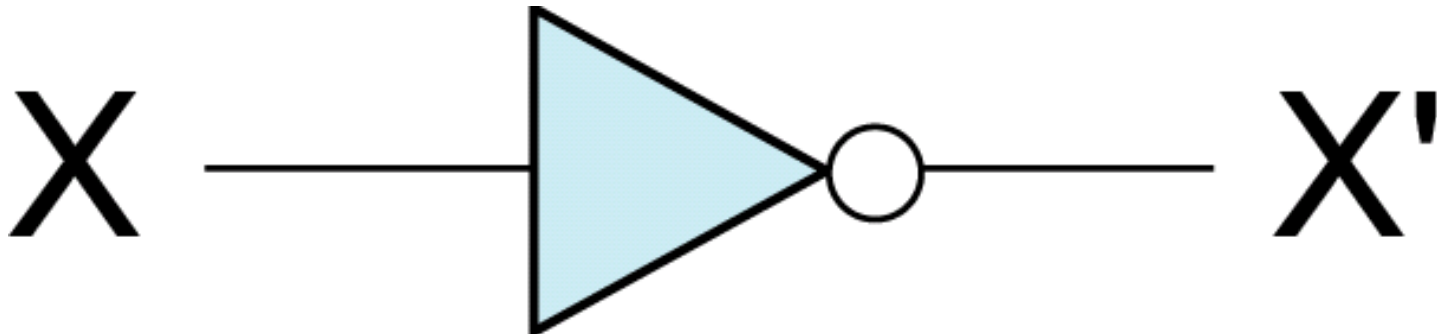
- 2.1 Introduction
 - 2.2 Basic Operations
 - 2.3 Boolean Expressions and Truth Tables
 - 2.4 Basic Theorems
 - 2.5 Commutative, Associative, and Distributive Laws
 - 2.6 Simplification Theorems
 - 2.7 Multiplying Out and Factoring
 - 2.8 DeMorgan's Laws
- Problems
- Laws and Theorems of Boolean Algebra



The electronic circuit which forms the inverse of X is referred to as an inverter

$$X' = 1 \text{ if } X = 0$$

$$X' = 0 \text{ if } X = 1$$

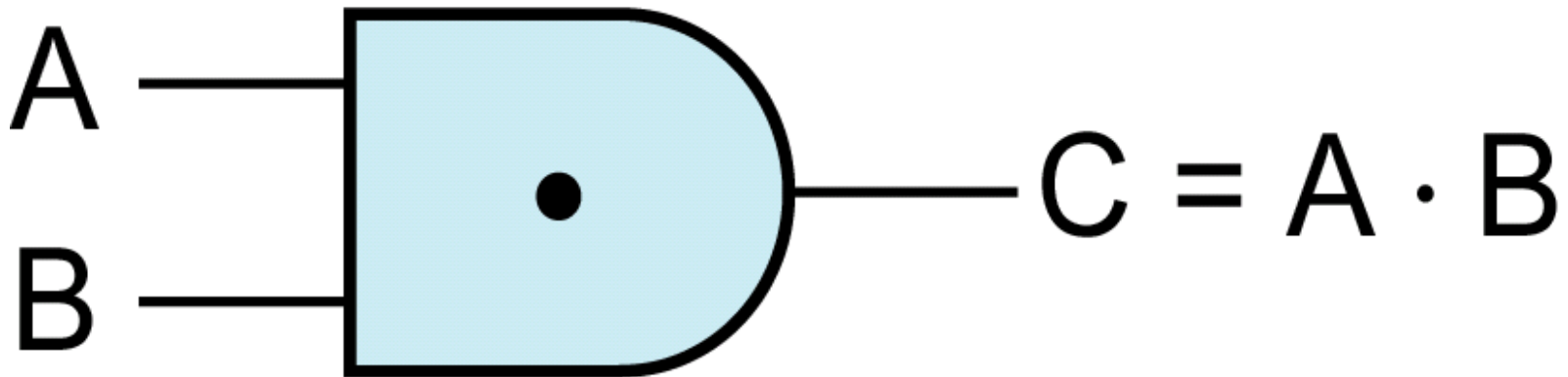


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AND Gate

Note that $C = 1$ if and only if A and B are both 1.

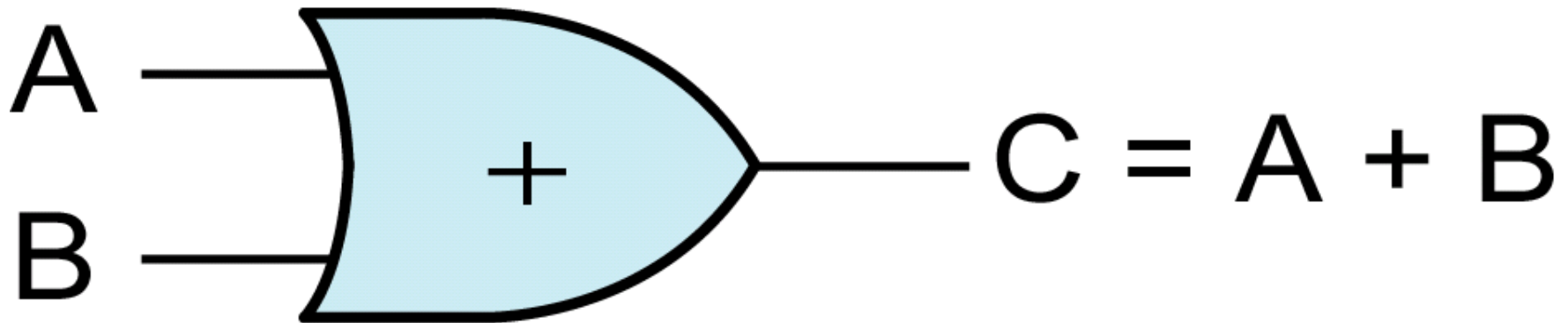
A	B	$C = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1



OR Gate

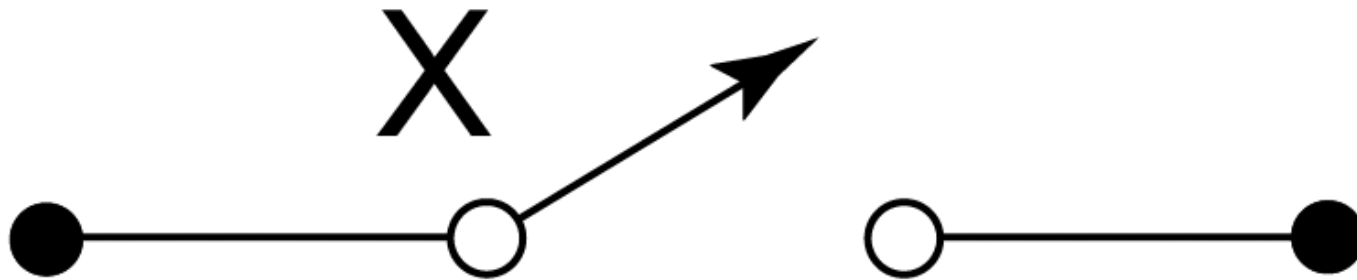
Note that $C = 1$ if and only if A or B (or both) are 1.

A	B	C = A + B
0	0	0
0	1	1
1	0	1
1	1	1



Switches

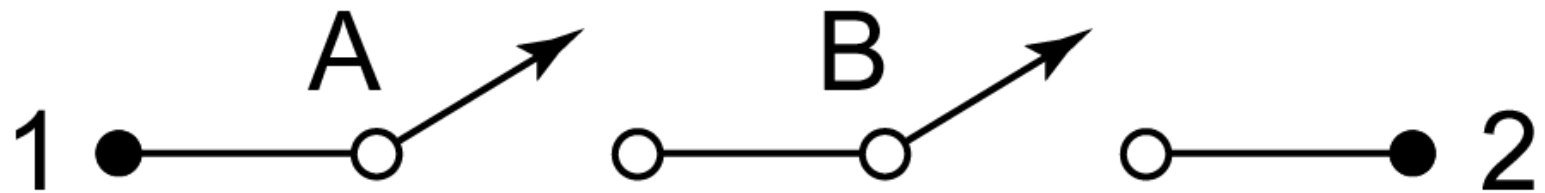
If switch X is open, then we will define the value of X to be 0; if switch X is closed, then we will define the value of X to be 1.



$X = 0$ switch open

$X = 1$ switch closed

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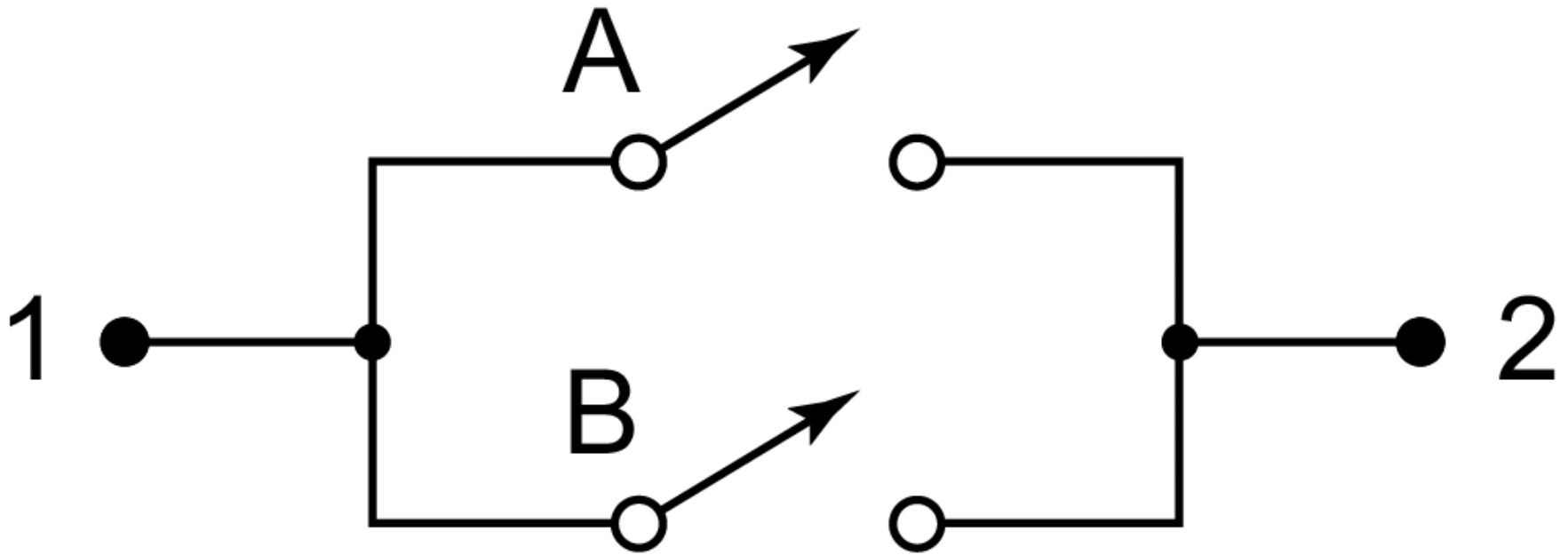


$T = 0$ open circuit between terminals 1 and 2

$T = 1$ closed circuit between terminals 1 and 2

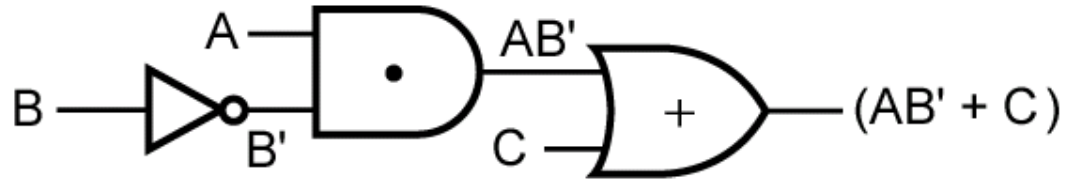
$$T = AB$$

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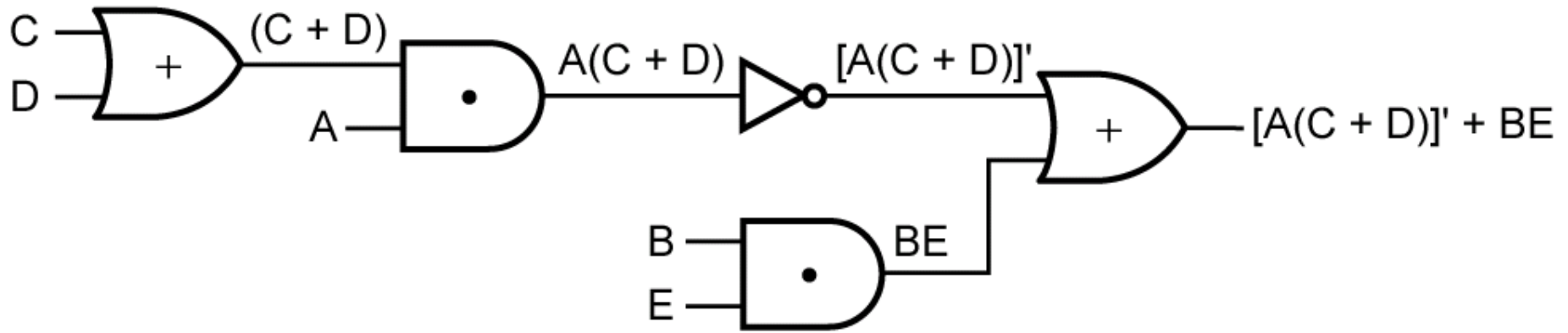


$$T = A + B$$

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(a) $F = AB' + C$



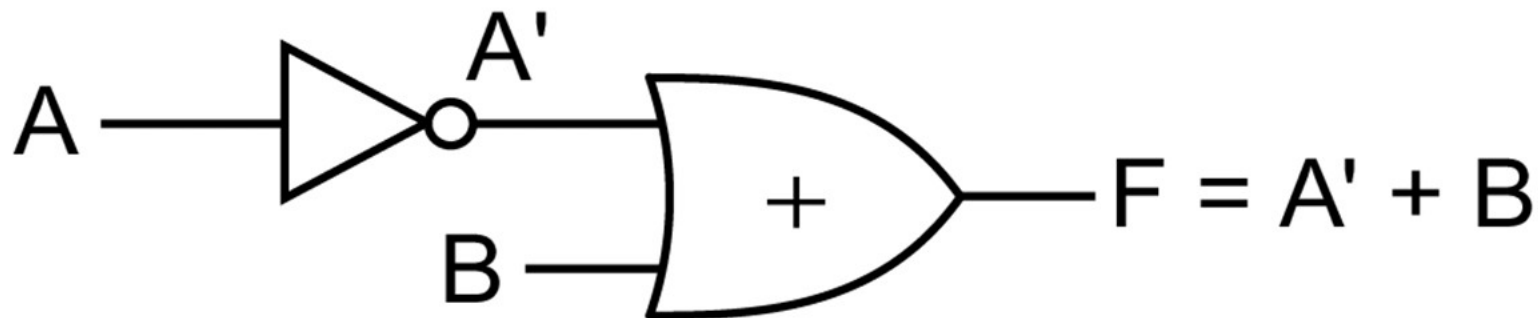
(b) $F = [A(C + D)]' + BE$

Figure 2-1: Circuits for Expressions (2-1) and (2-2)

Figure 2-2(b) shows a truth table which specifies the output of the circuit in Figure 2-2(a) for all possible combinations of values of the inputs A and B.

(b)

A	B	A'	$F = A' + B$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1



(a)

Figure 2-2: 2-Input Circuit

Since the expression $(A + C)(B' + C)$ has the same value as $AB' + C$ for all eight combinations of values of the variables A , B , and C , we conclude that:

$$AB' + C = (A + C)(B' + C) \quad (2-3)$$

A	B	C	B'	AB'	AB'+C	A+C	B'+C	(A+C)(B'+C)
0	0	0	1	0	0	0	1	0
0	0	1	1	0	1	1	1	1
0	1	0	0	0	0	0	0	0
0	1	1	0	0	1	1	1	1
1	0	0	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1
1	1	0	0	0	0	1	0	0
1	1	1	0	0	1	1	1	1

Table 2-1: Truth Table for 3 variables

Basic Theorems

The following basic laws and theorems of boolean algebra involve only a single variable:

Operations with 0 and 1:

$$X + 0 = X \quad (2-4)$$

$$X \cdot 1 = X \quad (2-4D)$$

$$X + 1 = 1 \quad (2-5)$$

$$X \cdot 0 = 0 \quad (2-5D)$$

Idempotent laws

$$X + X = X \quad (2-6)$$

$$X \cdot X = X \quad (2-6D)$$

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Involution law

$$(X')' = X \quad (2-7)$$

Laws of complementarity

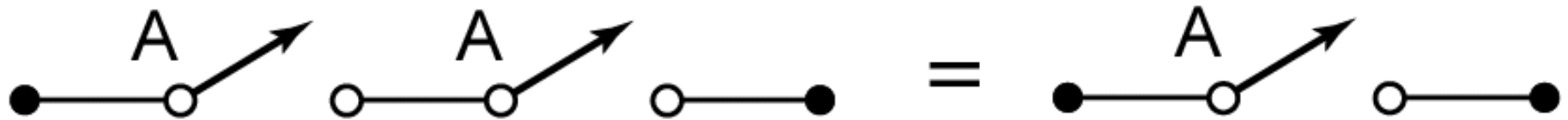
$$X + X' = 1 \quad (2-8) \quad X \cdot X' = 0 \quad (2-8D)$$

Each of these theorems is easily proved by showing that it is valid for both of the possible values of X . For example. To prove $X + X' = 1$, we observe that if

$$X = 0, \quad 0 + 0' = 0 + 1 = 1,$$

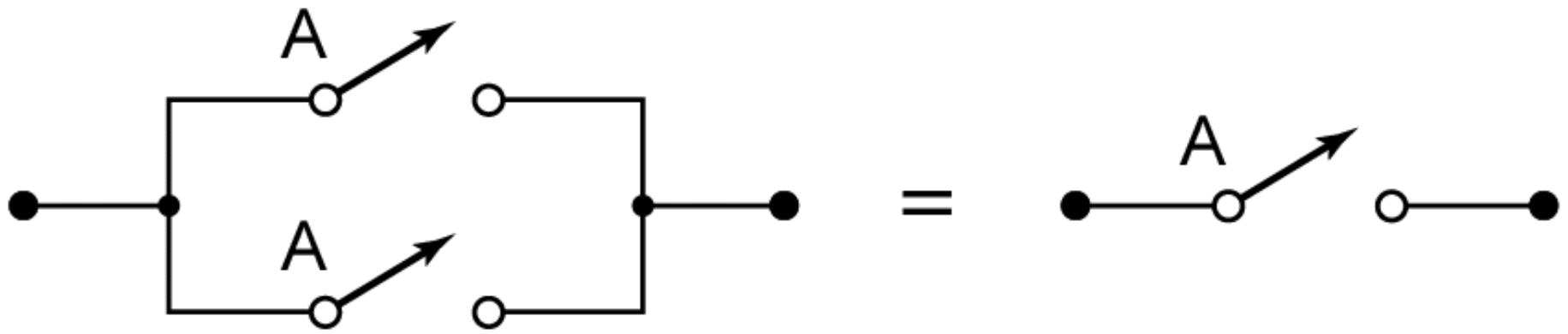
$$\text{and if } X = 1, \quad 1 + 1' = 1 + 0 = 1$$

If two switches are both labeled with the variable A , this means that both switches are open when $A = 0$ and both are closed when $A = 1$, thus the following circuits are equivalent:



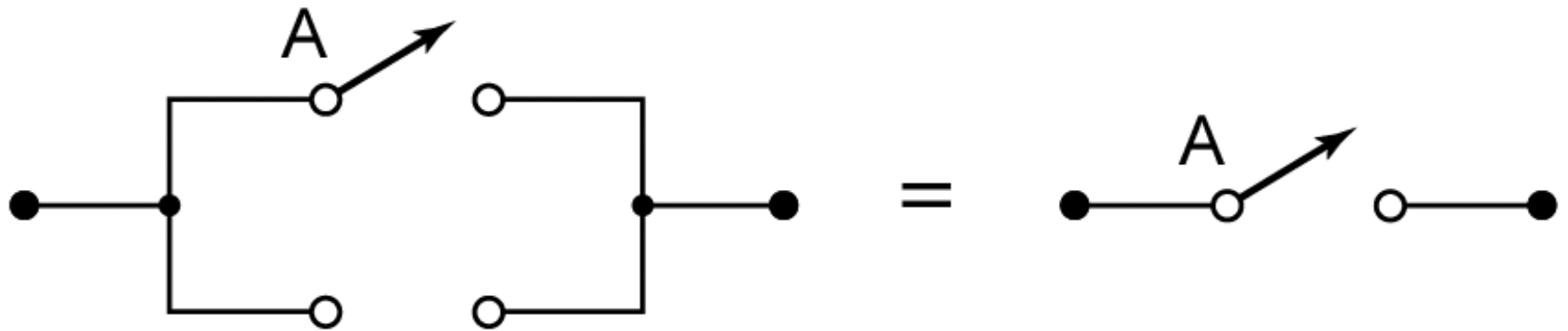
$$(A \cdot A = A)$$

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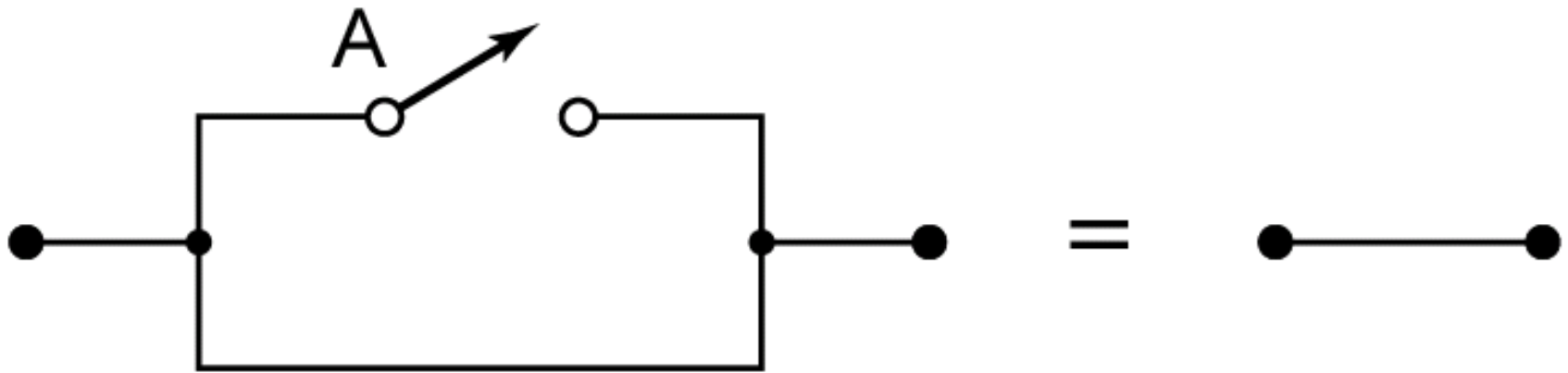


$$(A + A = A)$$

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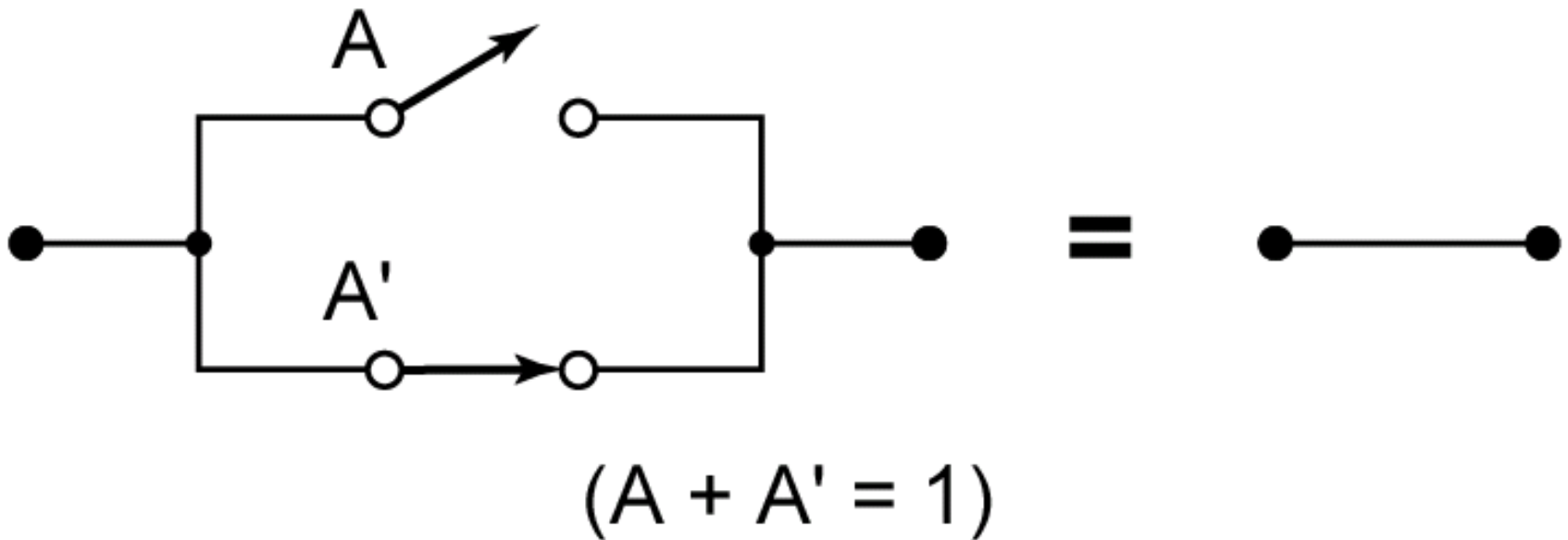


$$(A + 0 = A)$$



$$(A + 1 = 1)$$

A in parallel with A' can be replaced with a closed circuit because one or the other of the two switches is always closed.



Similarly, switch A in series with A' can be replaced with an open circuit because one or the other of the two switches is always open.



$$(A \cdot A' = 0)$$

Commutative and Associative Laws

Many of the laws of ordinary algebra, such as commutative and associative laws, also apply to Boolean algebra. The commutative laws for AND and OR, which follow directly from the definitions of the AND and OR operations, are

$$XY = YX \quad (2-9) \qquad X + Y = Y + X \quad (2-9D)$$

This means that the order in which the variables are written will not affect the result of applying the AND and OR operations. The associate laws also apply to AND and OR:

$$(XY)Z = X(YZ) = XYZ \quad (2-10)$$

$$(X + Y) + Z = X + (Y + Z) = X + Y + Z \quad (2-10D)$$

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Table 2-2: Proof of Associative Law for AND

X	Y	Z	XY	YZ	$(XY)Z$	$X(YZ)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Two 2-input **AND** gates can be replaced with a single 3-input **AND** gate (a). Similarly, two 2-input **OR** gates can be replaced with a single 3-input **OR** gate (b).

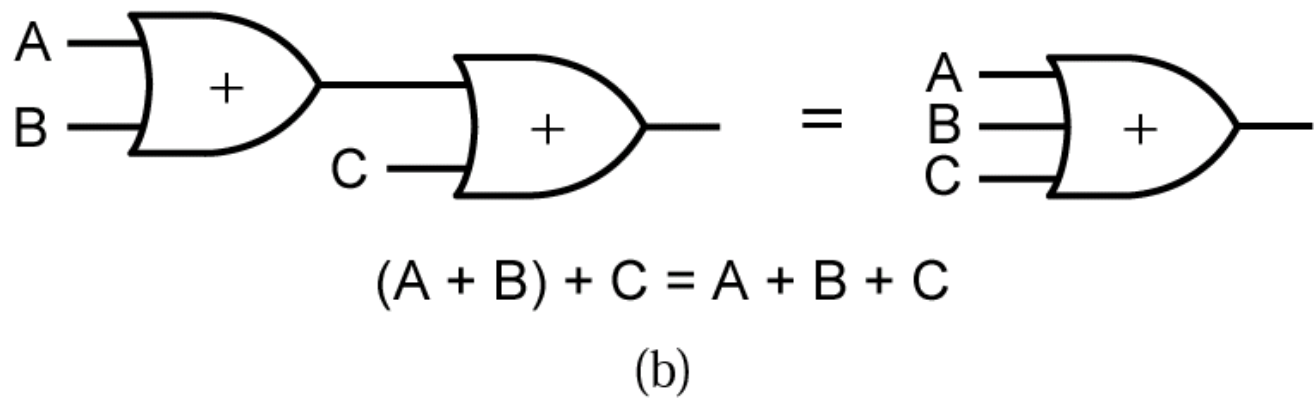
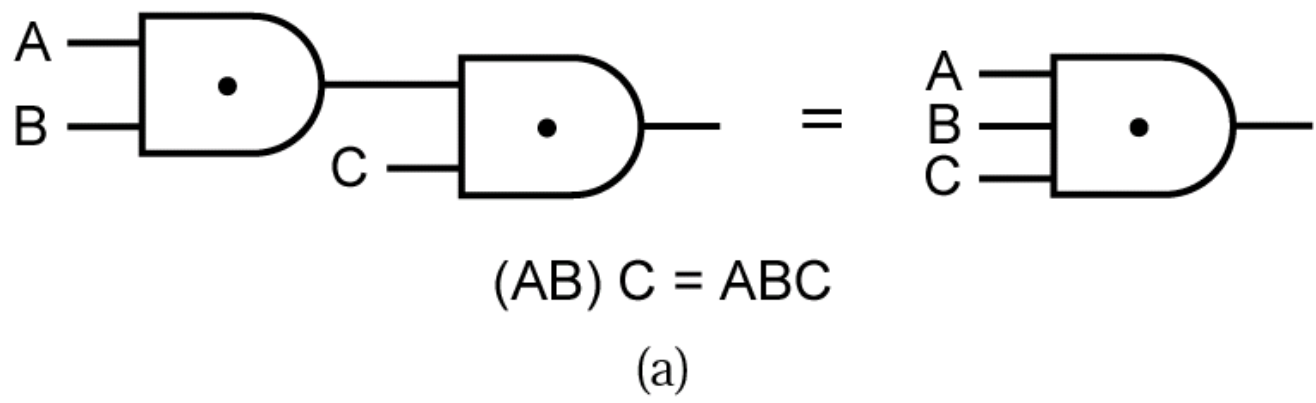


Figure 2-3: Associative Law for AND and OR

Distributive Law

Using a truth table, it is easy to show that the distributive law is valid:

$$X(Y + Z) = XY + XZ \quad (2-11)$$

In addition to the ordinary distributive law, a second distributive law is valid for Boolean algebra, but not for ordinary algebra:

$$X + YZ = (X + Y)(X + Z) \quad (2-11D)$$

Proof of the second distributive law follows:

$$\begin{aligned} (X + Y)(X + Z) &= X(X + Z) + Y(X + Z) = XX + XZ + YX + YZ && \text{(by (2-11))} \\ &= X + XZ + XY + YZ = X \cdot 1 + XZ + XY + YZ && \text{(by (2-6D) and (2-4D))} \\ &= X(1 + Z + Y) + YZ = X \cdot 1 + YZ = X + YZ && \text{(by (2-11), (2-5), and (2-4D))} \end{aligned}$$

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Simplification Theorems

The following theorems are useful in simplifying Boolean expressions:

$$XY + XY' = X \quad (2-12) \qquad (X + Y)(X + Y') = X \quad (2-12D)$$

$$X + XY = X \quad (2-13) \qquad X(X + Y) = X \quad (2-13D)$$

$$(X + Y')Y = XY \quad (2-14) \qquad XY' + Y = X + Y \quad (2-14D)$$

Proof of (2-13): $X + XY = X \cdot 1 + XY = X(1 + Y) = X \cdot 1 = X$

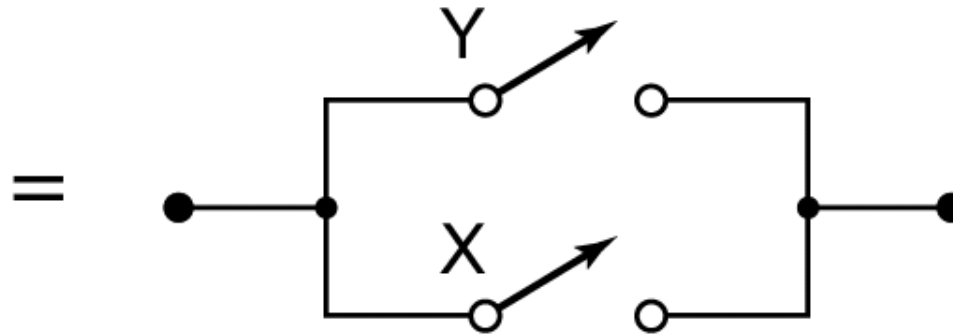
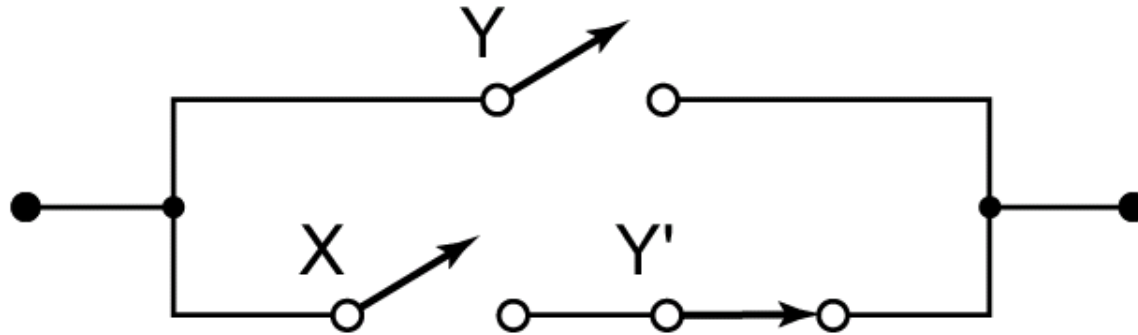
Proof of (2-13D): $X(X + Y) = XX + XY = X + XY = X$
(by (2-6D) and (2-13))

Proof of (2-14D): $Y + XY' = (Y + X)(Y + Y') = (Y + X)1 = Y + X$
(by (2-11 D) and (2-8))

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Illustration of

Theorem (2-14D): $XY' + Y = X + Y$

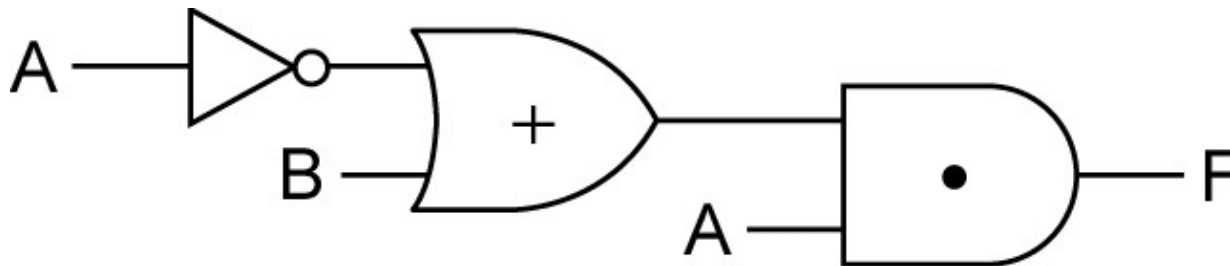


$$(Y + XY' = Y + X)$$

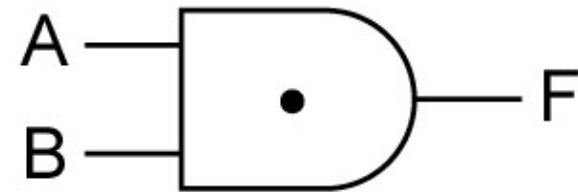
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$$F = A(A' + B)$$

By **Theorem (2-14)**, $(X + Y')$ = XY , the expression F simplifies to AB .



(a)



(b)

Figure 2-4: Equivalent Gate Circuits

Example 1

Simplify $Z = A'BC + A'$

This expression has the same form as (2-13) if we let $X = A'$ and $Y = BC$. Therefore, the expression simplifies to $Z = X + XY = X = A'$.

Example 2

Simplify $Z = \underbrace{[A + B'C + D + EF]}_X \underbrace{[A + B'C + (D + EF)']}_Y$

Substituting: $Z = [X + Y][X + Y']$

Then, by (2-12D), the expression reduces to

$$Z = X = A + B'C$$

Simplify (p. 43-44)

Example 3

Simplify $Z = \underbrace{(AB + C)}_{Y'} \underbrace{(B'D + C'E')}_X + \underbrace{(AB + C)'}_Y$

Substituting: $Z = Y' X + Y$

By, (2-14D): $Z = X + Y = B'D + C'E' + (AB + C)'$

Note that in this example we let $Y = (AB + C)'$ rather than $(AB + C)$ in order to match the form of (2-14D).

Simplify (p. 43-44)

Sum-Of-Products

An expression is said to be in *sum-of-products* (SOP) form when all products are the products of single variables. This form is the end result when an expression is fully multiplied out.

For example:

$$AB' + CD'E + AC'E' \quad (2-15)$$

$$ABC' + DEFG + H \quad (2-16)$$

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Product-Of-Sums

An expression is in *product-of-sums* (POS) form when all sums are the sums of single variables. It is usually easy to recognize a product-of-sums expression since it consists of a product of sum terms.

For example:

$$(A + B')(C + D' + E)(A + C' + E') \quad (2-18)$$

$$(A + B)(C + D + E)F \quad (2-19)$$

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EXAMPLE 1: Factor $A + B'CD$. This is of the form $X + YZ$
where $X = A$, $Y = B'$, and $Z = CD$, so
 $A + B'CD = (X + Y)(X + Z) = (A + B')(A + CD)$

$A + CD$ can be factored again using the second distributive law, so
 $A + B'CD = (A + B')(A + C)(A + D)$

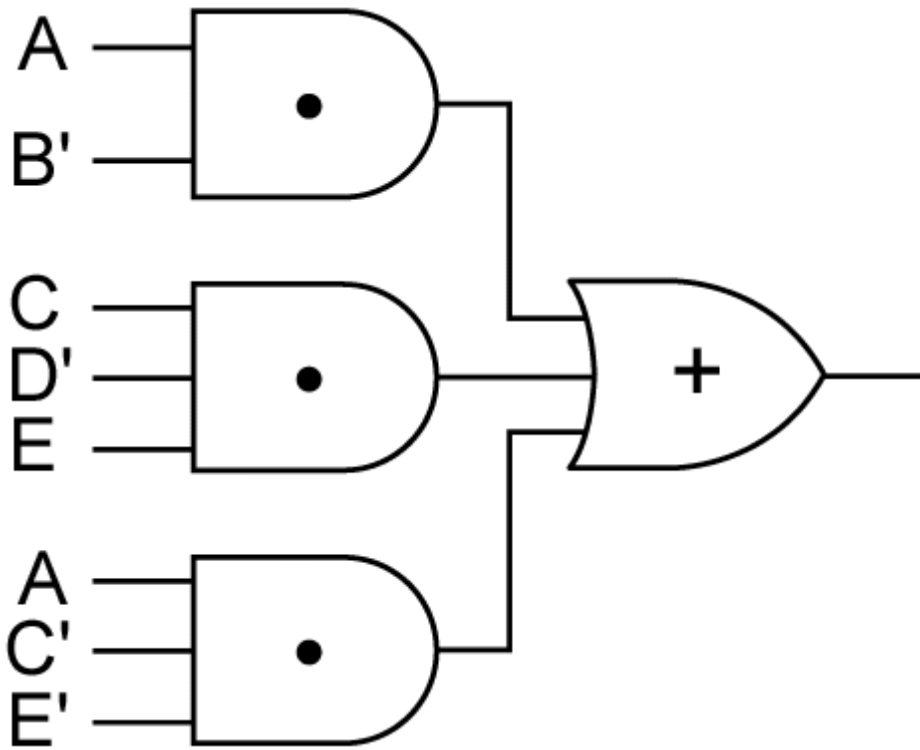
EXAMPLE 2: Factor $AB' + C'D$

$$\begin{aligned} AB' + C'D &= (AB' + C')(AB' + D) \\ &= (A + C')(B' + C')(A + D)(B' + D) \end{aligned}$$

EXAMPLE 3: Factor $C'D + C'E' + G'H$

$$\begin{aligned} C'D + C'E' + G'H &= C'(D + E') + G'H \\ &= (C' + G'H)(D + E' + G'H) \\ &= (C' + G')(C' + H)(D + E' + G')(D + E' + H) \end{aligned}$$

Factor (p. 45-46)



(2-15):

$$AB' + CD'E + AC'E'$$

(2-17):

$$A + B' + C + D'E$$

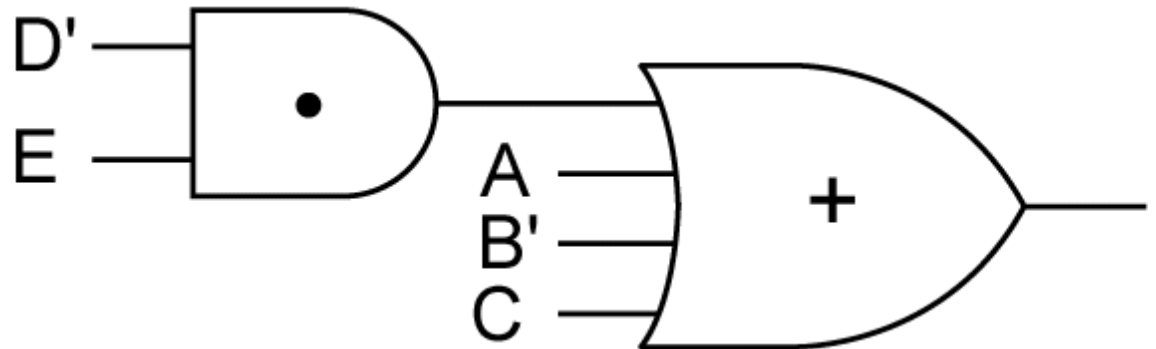
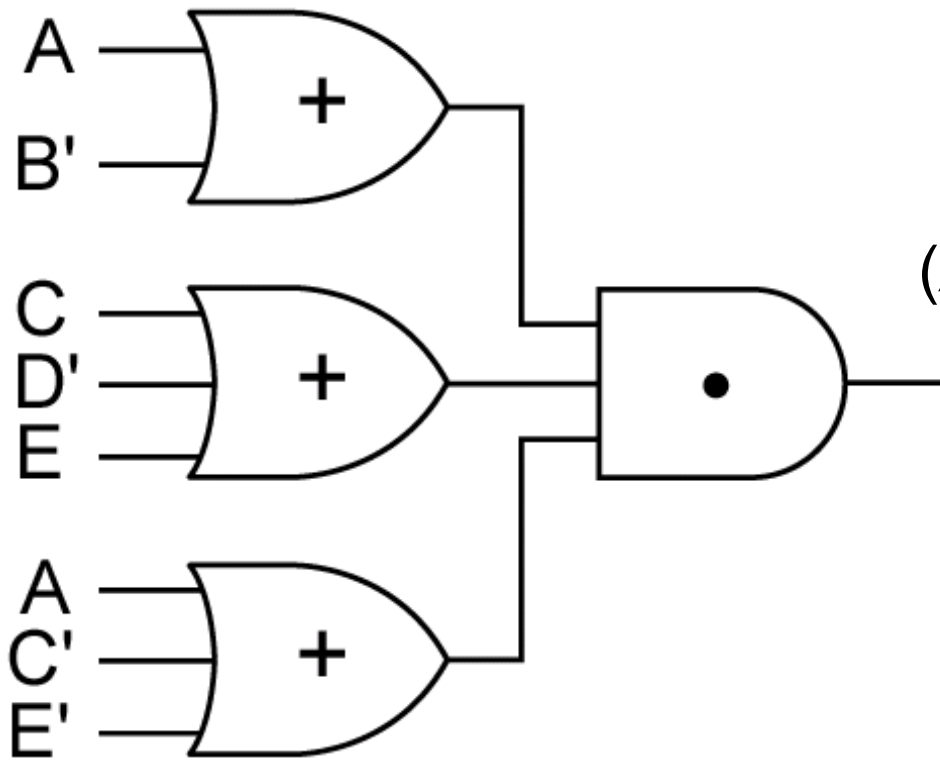


Figure 2-5: Circuits for Equations (2-15) and (2-17)



(2-18):

$$(A + B')(C + D' + E)(A + C' + E')$$

(2-20):

$$AB'C(D' + E)$$

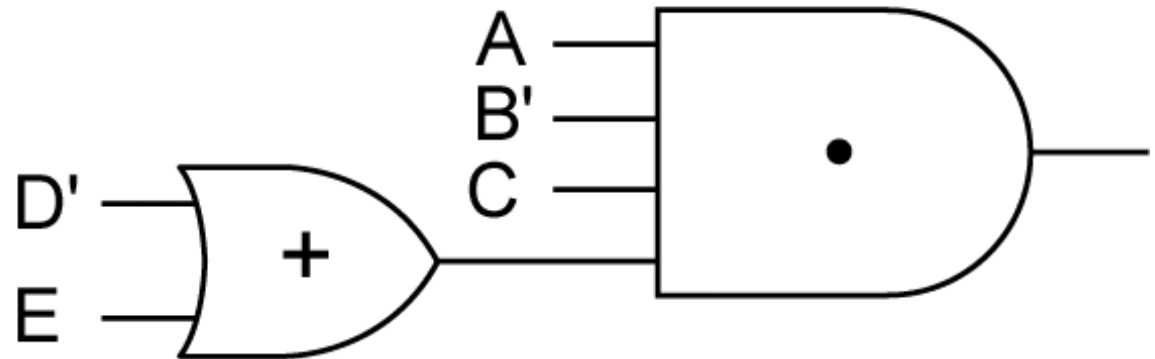


Figure 2-6: Circuits for Equations (2-18) and (2-20)

DeMorgan's Laws

$$(X + Y)' = X' Y' \quad (2-21)$$

$$(XY)' = X' + Y' \quad (2-22)$$

DeMorgan's laws are easily generalized to n variables:

$$(X_1 + X_2 + X_3 + \dots + X_n)' = X_1' X_2' X_3' \dots X_n' \quad (2-23)$$

$$(X_1 X_2 X_3 \dots X_n)' = X_1' + X_2' + X_3' + \dots + X_n' \quad (2-24)$$

For example, for $n = 3$,

$$(X_1 + X_2 + X_3)' = (X_1 + X_2)' X_3' = X_1' X_2' X_3'$$

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Example 1

To find the complement of $(A' + B)C'$, first apply (2-22) and then (2-21).

$$[(A' + B)C']' = (A' + B)' + (C')' = AB' + C$$

$$\begin{aligned} [(AB' + C)D' + E]' &= [(AB' + C)D']'E' && \text{(by (2-21))} \\ &= [(AB' + C)' + D]E' && \text{(by (2-22))} \\ &= [(AB')'C' + D]E' && \text{(by (2-21))} \\ &= [(A' + B)C' + D]E' && \text{(by (2-22))} \quad (2-25) \end{aligned}$$

Example 2

Note that in the final expressions, the complement operation is applied only to single variables.

Duality

Given an expression, the *dual* is formed by replacing **AND** with **OR**, **OR** with **AND**, 0 with 1, and 1 with 0. Variables and complements are left unchanged. The dual of **AND** is **OR** and the dual of **OR** is **AND**:

$$(XYZ\dots)^D = X + Y + Z + \dots \quad (X + Y + Z + \dots)^D = XYZ\dots \quad (2-26)$$

The dual of an expression may be found by complementing the entire expression and then complementing each individual variables. For example:

$$(AB' + C)' = (AB')'C' = (A' + B)C', \quad \text{so} \quad (AB' + C)^D = (A + B')C$$

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LAWS AND THEOREMS (a)

p. 55

Operations with 0 and 1:

$$1. X + 0 = X$$

$$1D. X \cdot 1 = X$$

$$2. X + 1 = 1$$

$$2D. X \cdot 0 = 0$$

Idempotent laws:

$$3. X + X = X$$

$$3D. X \cdot X = X$$

Involution law:

$$4. (X')' = X$$

Laws of complementarity:

$$5. X + X' = 1$$

$$5D. X \cdot X' = 0$$

LAWS AND THEOREMS (b)

p. 55

Commutative laws:

$$6. X + Y = Y + X$$

$$6D. XY = YX$$

Associative laws:

$$7. (X + Y) + Z = X + (Y + Z) \\ = X + Y + Z$$

$$7D. (XY)Z = X(YZ) = XYZ$$

Distributive laws:

$$8. X(Y + Z) = XY + XZ$$

$$8D. X + YZ = (X + Y)(X + Z)$$

Simplification theorems:

$$9. XY + XY' = X$$

$$9D. (X + Y)(X + Y') = X$$

$$10. X + XY = X$$

$$10D. X(X + Y) = X$$

$$11. (X + Y')Y = XY$$

$$11D. XY' + Y = X + Y$$

LAWS AND THEOREMS (c)

p. 55

DeMorgan's laws:

$$12. (X + Y + Z + \dots)' = X'Y'Z' \dots$$

$$12D. (XYZ \dots)' = X' + Y' + Z' + \dots$$

Duality:

$$13. (X + Y + Z + \dots)^D = XYZ \dots$$

$$13D. (XYZ \dots)^D = X + Y + Z + \dots$$

Theorem for multiplying out and factoring:

$$14. (X + Y)(X' + Z) = XZ + X'Y$$

$$14D. XY + X'Z = (X + Z)(X' + Y)$$

Consensus theorem:

$$15. XY + YZ + X'Z = XY + X'Z$$

$$15D. (X + Y)(Y + Z)(X' + Z) = (X + Y)(X' + Z)$$