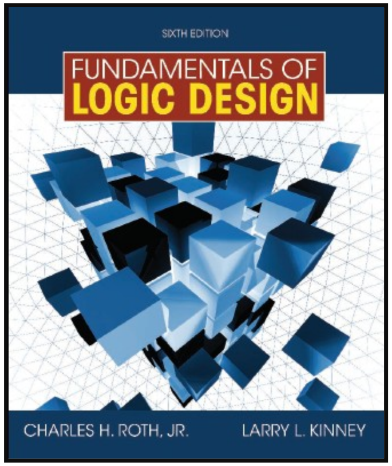
CHAPTER 3

BOOLEAN ALGEBRA
(continued)



This chapter in the book includes:

Distributive Laws

Given an expression in product-of-sums form, the corresponding sum-of-products expression can be obtained by multiplying out, using the two distributive laws:

$$X(Y + Z) = XY + XZ \qquad (3\text{-}1)$$

$$(X + Y)(X + Z) = X + YZ \qquad (3\text{-}2)$$

In addition, the following theorem is very useful for factoring and multiplying out:

$$(X + Y)(X' + Z) = XZ + X'Y \qquad (3\text{-}3)$$

In the following example, if we were to multiply out by brute force, we would generate 162 terms, and 158 of these terms would then have to be eliminated to simplify the expression. Instead, we will use the distributive laws to simplify the process.

$$
\begin{aligned}
&(A + B + C')(A + B + D)(A + B + E)\overbrace{(A + D' + E)(A' + C)} \\
&= (A + B + C'D)(A + B + E)[AC + A'(D' + E)] \\
&= (A + B + C'DE)(AC + A'D' + A'E) \\
&= AC + \cancel{ABC} + A'BD' + A'BE + A'C'DE
\end{aligned}
$$

Example (3-4), p. 63

The same theorems that are useful for multiplying out expressions are useful for factoring. By repeatedly applying (3-1), (3-2), and (3-3), any expression can be converted to a product-of-sums form.

$$AC + A'BD' + A'BE + A'C'DE$$
$$= \underbrace{AC}_{XZ} + \underbrace{A'}_{X'}(\underbrace{BD' + BE + C'DE}_{Y})$$
$$= (A + BD' + BE + C'DE)(A' + C)$$
$$= [\underbrace{A + C'DE}_{X} + \underbrace{B}_{Y}(\underbrace{D' + E}_{Z})](A' + C)$$
$$= (A + B + C'DE)(A + \cancel{C'DE} + D' + E)(A' + C)$$
$$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C) \quad (3\text{-}5)$$

Exclusive-OR and Equivalence Operations

The *exclusive-OR* operation ($\oplus$) is defined as follows:

$$0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$$

$$1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$$

The *equivalence* operation ($\equiv$) is defined by:

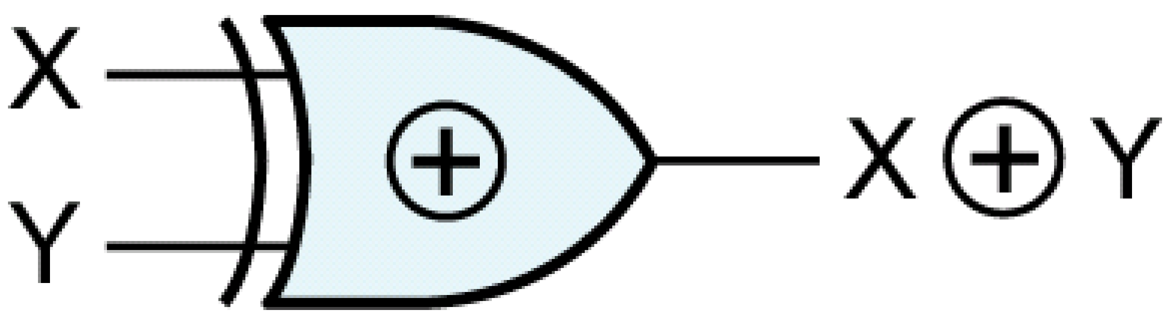
$$(0 \equiv 0) = 1 \qquad (0 \equiv 1) = 0$$

$$(1 \equiv 0) = 0 \qquad (1 \equiv 1) = 1$$

$$X \oplus Y = X'Y + XY'$$

X	Y	$X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

We will use the following symbol for an **exclusive-OR** gate:



Section 3.2, p. 64

The following theorems apply to exclusive OR:

$$X \oplus 0 = X \tag{3-8}$$

$$X \oplus 1 = X' \tag{3-9}$$

$$X \oplus X = 0 \tag{3-10}$$

$$X \oplus X' = 1 \tag{3-11}$$

$$X \oplus Y = Y \oplus X \text{ (commutative law)} \tag{3-12}$$

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z \text{ (associative law)} \tag{3-13}$$

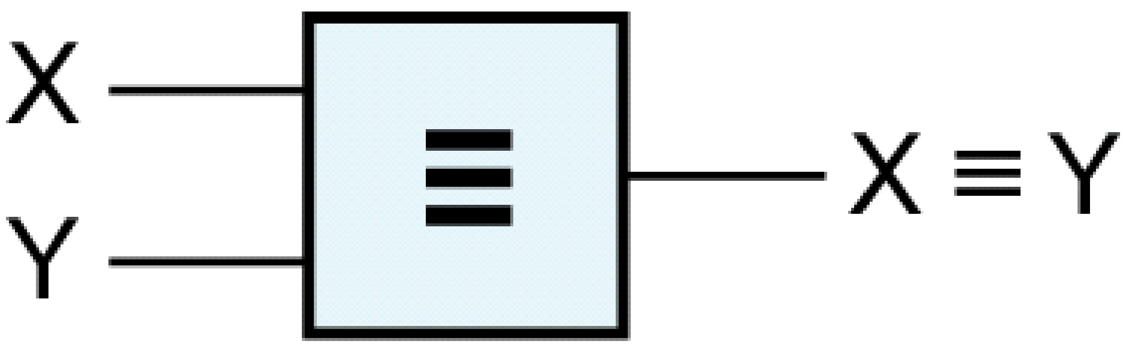
$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive law)} \tag{3-14}$$

$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y' \tag{3-15}$$

$$(X \equiv Y) = XY + X'Y'$$

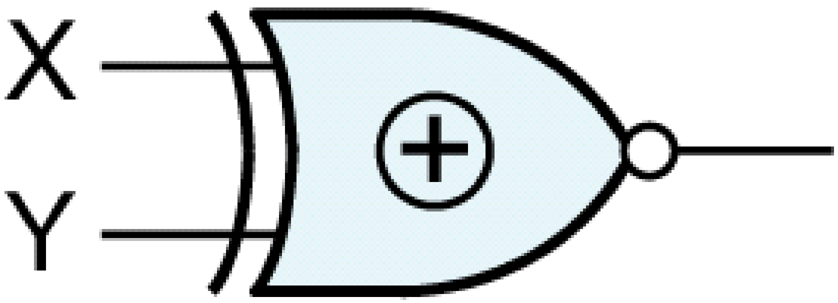
X	Y	$X \equiv Y$
0	0	1
0	1	0
1	0	0
1	1	1

We will use the following symbol for an **equivalence** gate:



Section 3.2, p. 65

Because equivalence is the complement of **exclusive-OR**, an alternate symbol of the equivalence gate is an **exclusive-OR** gate with a complemented output:



$$(X \oplus Y)' = (X \equiv Y)$$

The equivalence gate is also called an **exclusive-NOR** gate.

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Example 1:

$$F = (A'B \equiv C) + (B \oplus AC')$$

By (3-6) and (3-17),

$F = [(A'B)C + (A'B)'C'] + [B'(AC') + B(AC')']$
$= A'BC + (A + B')C' + AB'C' + B(A' + C)$
$= B(A'C + A' + C) + C'(A + B' + AB') = B(A' + C) + C'(A + B')$

Example 2:

$$A' \oplus B \oplus C = [A'B' + (A')'B] \oplus C$$

$= (A'B' + AB)C' + (A'B' + AB)'C$ (by (3-6))
$= (A'B' + AB)C' + (A'B + AB')C$ (by (3-19))
$= A'B'C' + ABC' + A'BC + AB'C$

Section 3.2 (p. 66)

The Consensus Theorem

The consensus theorem can be stated as follows:

$XY + X'Z + YZ = XY + X'Z$ **(3-20)**

Dual Form:

$(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$ **(3-21)**

Section 3.2 (p. 66-67)

Consensus Theorem Proof

$$XY + X'Z + YZ = XY + X'Z + (X + X')YZ$$
$$= (XY + XYZ) + (X'Z + X'YZ)$$
$$= XY(1 + Z) + X'Z(1 + Y) = XY + X'Z$$

Section 3.3 (p. 67)

Basic methods for simplifying functions

1. *Combining terms.* Use the theorem $XY + XY' = X$ to combine two terms. For example,

$$abc'd' + abcd' = abd' \qquad [X = abd',\ Y = c] \qquad (3\text{-}24)$$

2. *Eliminating terms.* Use the theorem $X + XY = X$ to eliminate redundant terms if possible; then try to apply the consensus theorem ($XY + X'Z + YZ = XY + X'Z$) to eliminate any consensus terms. For example,

$$a'b + a'bc = a'b \qquad [X = a'b]$$

$$a'bc' + bcd + a'bd = a'bc' + bcd \qquad [X = c,\ Y = bd,\ Z = a'b] \quad (3\text{-}24)$$

Section 3.4 (p. 68-69)

3. *Eliminating literals.* Use the theorem $X + X'Y = X + Y$ to eliminate redundant literals. Simple factoring may be necessary before the theorem is applied.

$$
\begin{aligned}
A'B + A'B'C'D' + ABCD' &= A'(B + B'C'D') + ABCD' \\
&= A'(B + C'D') + ABCD' \\
&= B(A' + ACD') + A'C'D' \\
&= B(A' + CD') + A'C'D' \\
&= A'B + BCD' + A'C'D' \qquad \text{(3-26)}
\end{aligned}
$$

4. *Adding redundant terms.* Redundant terms can be introduced in several ways such as adding xx', multiplying by $(x + x')$, adding yz to $xy + x'z$, or adding xy to x. When possible, the added terms should be chosen so that they will combine with or eliminate other terms.

$WX + XY + X'Z' + WY'Z'$ (add WZ' by consensus theorem)

$= WX + XY + X'Z' + WY'Z' + WZ'$ (eliminate $WY'Z'$)

$= WX + XY + X'Z' + WZ'$ (eliminate WZ')

$= WX + XY + X'Z'$ (3-27)

The following comprehensive example illustrates use of all four methods:

$$\underbrace{A'B'C'D' + A'BC'D'}_{①\ A'C'D'} + A'BD + \underbrace{\cancel{A'BC'D}}_{②} + ABCD + ACD' + B'CD'$$

$$= A'C'D' + BD(A' \underset{③}{+} AC) + ACD' + B'CD'$$

$$= A'C'D' + A'BD + \underbrace{BCD + ACD'}_{+ ABC\ ④} + B'CD'$$

$$= A'C'D' + \underbrace{A'BD + \cancel{BCD} + \cancel{ACD'} + \overbrace{B'CD' + ABC}^{\text{consensus } ACD'}}_{\text{consensus } BCD}$$

$$= A'C'D' + A'BD + B'CD' + ABC$$

Example (3-28), p 69-70

Proving Validity of an Equation

Often we will need to determine if an equation is valid for all combinations of values of the variables. Several methods can be used to determine if an equation is valid:

1. Construct a truth table and evaluate both sides of the equation for all combinations of values of the variables. (This method is rather tedious if the number of variables is large, and it certainly is not very elegant.)

2. Manipulate one side of the equation by applying various theorems until it is identical with the other side.

3. Reduce both sides of the equation independently to the same expression.

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4. It is permissible to perform the same operation on both sides of the equation provided that the operation is reversible. For example, it is all right to complement both sides of the equation, but it is *not* permissible to multiply both sides of the equation by the same expression. (Multiplication is not reversible because division is not defined for Boolean algebra.) Similarly, it is *not* permissible to add the same term to both sides of the equation because subtraction is not defined for Boolean algebra.

To prove that an equation is *not* valid, it is sufficient to show one combination of values of the variables for which the two sides of the equation have different values. When using method 2 or 3 above to prove that an equation is valid, a useful strategy is to

1. First reduce both sides to a sum of products (or a product of sums).
2. Compare the two sides of the equation to see how they differ.
3. Then try to add terms to one side of the equation that are present on the other side.
4. Finally try to eliminate terms from one side that are not present on the other.

Whatever method is used, frequently compare both sides of the equation and let the different between them serve as a guide for what steps to take next.

Example: Show that

$A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC$

Solution: Starting with the left side,

$$A'BD' + BCD + ABC' + AB'D$$

$$= A'BD' + BCD + ABC' + AB'D + BC'D' + A'BC + ABD$$

(add consensus of $A'BD'$ and ABC')

(add consensus of $A'BD'$ and BCD)

(add consensus of BCD and ABC')

$$= AD + A'BD' + BCD + ABC' + BC'D' + A'BC = BC'D' + AD + A'BC$$

(eliminate consensus of $BC'D'$ and AD)

(eliminate consensus of AD and $A'BC$)

(eliminate consensus of $BC'D'$ and $A'BC$) (3-30)

Example 1 (p. 71)

Differences between Boolean algebra and ordinary algebra

As we have previously observed, some of the theorems of Boolean algebra are not true for ordinary algebra. Similarly, some of the theorems of ordinary algebra are *not* true for Boolean algebra. Consider, for example, the cancellation law for ordinary algebra:

$$\text{If } x + y = x + z, \qquad \text{then} \quad y = z \tag{3-31}$$

The cancellation law is *not* true for Boolean algebra. We will demonstrate this by constructing a counterexample in which $x + y = x + z$ but $y \neq z$. Let $x = 1$, $y = 0$, $z = 1$. Then,

$$1 + 0 = 1 + 1 \text{ but } 0 \neq 1$$

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In ordinary algebra, the cancellation law for multiplication is

$$\text{If } xy = xz, \quad \text{then} \quad y = z \qquad (3\text{-}32)$$

This law is valid provided $x \neq 0$.

In Boolean algebra, the cancellation law for multiplication is also *not* valid when $x = 0$. (Let $x = 0$, $y = 0$, $z = 1$; then $0 \cdot 0 = 0 \cdot 1$, but $0 \neq 1$). Because $x = 0$ about half the time in switching algebra, the cancellation law for multiplication cannot be used.

Similarities between Boolean algebra and ordinary algebra

Even though the statements in the previous 2 slides (3-31 and 3-32) are generally false for Boolean algebra, the converses are true:

If $y = z$, then $x + y = x + z$ (3-33)

If $y = z$, then $xy = xz$ (3-34)

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