CHAPTER 1

Vector Analysis

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1.1 Overview

Vector analysis provides an elegant mathematical language in which electromagnetic theory is conveniently expressed and best understood. In order for students to better understand electromagnetic principles, it is imperative for them to use this mathematical language fluently. Junior or senior level undergraduates may not have adequate knowledge about vector analysis for electromagnetic, although it is likely that vector concepts and operations are introduced in calculus courses.

We are going to deal with four major topics in vector analysis: (1) In Sections 1.2 to 1.4, we will discuss vector algebra, including vector addition, subtraction and multiplications; (2) In Sections 1.5, we will discuss vector representation and vector algebra in orthogonal coordinate systems, including Cartesian, cylindrical and spherical systems; and (3) In Sections 1.6 and 1.7, we will discuss vector calculus, which encompasses differentiation and integration of vectors; line, surface and volume integrals; the del (\( \nabla \)) operator; and the gradient, divergence and curl operations.

Although we are going to solve our examples in both traditional way (without Matlab) and contemporary way (with MATLAB), we still emphasize that, as a powerful mathematical tool, MATLAB is widely used in engineering curriculum and in industry. Also, vector analysis, which is so crucial in describing electromagnetic phenomena, can be easily implemented using MATLAB.

1.2 Scalars and Vectors

Quantities that can be described by a magnitude alone are called scalars. Energy, temperature, weight, and mass are all examples of scalar quantities. Other quantities, called vectors, require both a magnitude and a direction to fully characterize them. Examples of vector quantities include force, velocity, and acceleration. Thus, a car traveling at 30 miles per hour (mph) can be described by the scalar quantity speed. However, a car traveling 30 mph in a northwest direction can be described by the vector quantity velocity, which has both a magnitude (the 30 mph speed) and a direction (northwest).
In electromagnetics, we frequently use the concept of a field. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are **scalar fields** and **vector fields**. Temperature distribution on a printed circuit board and carbon dioxide distribution in the atmosphere are examples of scalar fields. Wind velocity distribution in California and gravitational force distribution in Rocky Mountains are examples of vector fields.

Please note that in this textbook, **boldface** type will be used to denote a vector, for example, \( \mathbf{A} \). Scalars are printed in italic type, for example, \( A \). Since it is difficult to write bold face letters by hand, it is popular to use an arrow or a bar over a letter (\( \bar{A} \) or \( \mathbf{A} \)) or use a bar below a letter (\( \overline{A} \)) to describe a handwritten vector, and a scalar is written without adding any arrows or bars.

### 1.3 Vector Addition and Subtraction

A vector has both magnitude and direction. If the magnitude of a vector \( \mathbf{A} \) is written as \( |\mathbf{A}| \) or \( A \), the direction of the vector can be specified by the dimensionless \textit{unit vector} \( \mathbf{a}_A \) defined by

\[
\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}
\]  

(1.1)

Since the \textit{unit vector} has unity magnitude

\[
|\mathbf{a}_A| = 1
\]  

(1.2)

and points in the same direction as \( \mathbf{A} \), we can specify \( \mathbf{A} \) in terms of its magnitude \( A \) and direction \( \mathbf{a}_A \) as

\[
\mathbf{A} = |\mathbf{A}| \mathbf{a}_A
\]  

(1.3)

**Figure 1-1** shows the vector \( \mathbf{A} \) represented by a straight line of length \( A \) with an arrow pointing in the direction \( \mathbf{a}_A \). If two vectors have the same magnitude and direction, we define them to be \textit{equal vectors}, even though they may be displaced in space.

Vector addition follows the parallelogram rule as shown in **Figure 1-2 (a)**, where the sum of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) gives another vector \( \mathbf{C} \) which lies along the diagonal of the parallelogram. The parallelogram rule is equivalent to the tip-to-tail rule as shown in **Figure 1-2 (b)**, where the tail of vector \( \mathbf{B} \) connects to the tip of vector \( \mathbf{A} \) and the sum vector \( \mathbf{C} \) connects the tail of \( \mathbf{A} \) to the head of \( \mathbf{B} \).
FIGURE 1–1
Graphical representation of a vector \( \mathbf{A} \) with magnitude \(|\mathbf{A}|\) and unit vector \( \mathbf{a}_A \).

FIGURE 1–2
Vector addition using (a) parallelogram rule and (b) tip-to-tail rule.

It’s easy to show that vector addition obeys the commutative, associative and distributive laws summarized as follows:

Commutative Law: \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \) \hspace{1cm} (1.4)
Associative Law: \( \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \) \hspace{1cm} (1.5)
Distributive law: \( k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B} \) \hspace{1cm} (1.6)

In (1.6), the multiplication of a vector by a scalar can be defined as

\[ k\mathbf{A} = kA \mathbf{a}_A \] \hspace{1cm} (1.7)

If \( k \) is a positive scalar, the magnitude of \( \mathbf{A} \) will be changed by \( k \) times without changing the direction.
Vector subtraction can be defined through vector addition as

\[ \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \]  

(1.8)

where \((-\mathbf{B})\) is the negative vector of \(\mathbf{B}\) which has the same magnitude as \(\mathbf{B}\) but is pointing in the opposite direction of \(\mathbf{B}\).

If we are not considering vector fields, we can add or subtract vectors at different positions in space. The ability to employ vector notation allows us the convenience of visualizing problems with or without the specification of a coordinate system. After choosing the coordinate system that most concisely describes the distribution of the field, we then specify the field with the components determined with regard to that coordinate system (i.e., Cartesian, cylindrical, and spherical). Detailed exposition of vector operations will be given in Cartesian (rectangular) coordinates with the equivalent results just stated in the other systems. A vector in Cartesian coordinate system can be specified by stating its three components. For example, vector \(\mathbf{A}\) can be expressed as

\[ \mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \]  

(1.9)

where \(A_x, A_y, \) and \(A_z\) are the magnitudes of the \(x, y,\) and \(z\) components of the vector \(\mathbf{A}\), respectively; and \(\mathbf{a}_x, \mathbf{a}_y, \) and \(\mathbf{a}_z\) are the unit vectors directed along the \(x, y,\) and \(z\) axes.

The addition of two vectors in Cartesian coordinates can be written as

\[ \mathbf{A} + \mathbf{B} = (A_x + B_x) \mathbf{a}_x + (A_y + B_y) \mathbf{a}_y + (A_z + B_z) \mathbf{a}_z \]  

(1.10)

Two vectors are equal \((\mathbf{A} = \mathbf{B})\) if and only if their corresponding components are equal. That is \(A_x = B_x, A_y = B_y\) and \(A_z = B_z\). It is noted that two vectors are equal does not mean they are necessarily identical. Two parallel vectors with the same magnitude and pointing in the same direction are equal, but their tip and tale points may not be the same. If they have the same tip and tale points (meaning that one vector exactly coincides with another vector), they are identical.

**EXAMPLE 1.1**

Given the vectors \(\mathbf{A} = 3\mathbf{a}_x\) and \(\mathbf{B} = 4\mathbf{a}_y\), compute the sum \(\mathbf{C} = \mathbf{A} + \mathbf{B}\). Find the magnitude of \(\mathbf{C}\) and the unit vector \(\mathbf{a}_C\). Plot and label these vectors and the unit vectors \(\mathbf{a}_x\) and \(\mathbf{a}_y\) to illustrate the “tip-to-tail” addition method.

**Solution:**

The sum is \(\mathbf{C} = 3\mathbf{a}_x + 4\mathbf{a}_y\).

The magnitude is

\[ |\mathbf{C}| = \sqrt{3^2 + 4^2} = 5 \]

and the unit vector is

\[ \mathbf{a}_C = \frac{(3\mathbf{a}_x + 4\mathbf{a}_y)}{\sqrt{3^2 + 4^2}} = 0.6\mathbf{a}_x + 0.8\mathbf{a}_y. \]

**MATLAB Solution:**

In MATLAB notation, the two-dimensional vector \(\mathbf{A}\) can be written in terms of its \(x\) and \(y\) components as
The second vector \( \mathbf{B} \) is written as
\[
\mathbf{B} = [0 \ 4];
\]
where we employ the semicolon in order to display no results.

Having “stored” the two vectors \( \mathbf{A} \) and \( \mathbf{B} \) into computer memory, we can then perform various mathematical operations. The vectors can be added as \( \mathbf{C} = \mathbf{A} + \mathbf{B} \) by typing
\[
\mathbf{C} = \mathbf{A} + \mathbf{B}
\]
\[
\mathbf{C} = [3 \ 4]
\]
The vector is interpreted as \( \mathbf{C} = 3\mathbf{a}_x + 4\mathbf{a}_y \).

The magnitude of \( \mathbf{C} \) can be computed using the MATLAB command \texttt{norm(C)}. The unit vector \( \mathbf{a}_C \) can be found using the norm function. It is equal to the vector divided by the magnitude of the vector. This is illustrated as follows:
\[
\texttt{magC = norm(C)}
\]
\[
\texttt{magC = 5}
\]
\[
\texttt{a}_C = \mathbf{C}/\texttt{magC}
\]
\[
\texttt{a}_C = [0.6000 \ 0.8000]
\]
MATLAB gives the result to the (user-controllable) default accuracy of four decimal places. At the present time, MATLAB does not have a feature to create a vector directly by drawing with arrows. However, thanks to Jeff Chang and Tom Davis, there exists a user contributed file entitled \texttt{arrow3} at \texttt{http://www.mathworks.com/matlabcentral/fileexchange/loadFile.do?objectId=1430}.
A title and captions have been added to the plot using MATLAB plot options. The MATLAB source code is listed in \texttt{ex101.m}. 

\[
\texttt{magC = norm(C)}
\]
\[
\texttt{magC = 5}
\]
\[
\texttt{a}_C = \mathbf{C}/\texttt{magC}
\]
\[
\texttt{a}_C = [0.6000 \ 0.8000]
\]
1.4 Vector Multiplication

The operation of multiplication on vectors can be carried out in two different ways, yielding two very different results. They are scalar (or dot) product and vector (or cross) product.

1.4.1 Scalar (or Dot) Product

The first vector multiplication operation is called either the scalar product, or the dot product. One definition of the scalar product of two vectors is

$$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}||\mathbf{B}|\cos \theta = AB \cos \theta$$

(1.11)

This multiplication results in a scalar product that is equal to the product of the magnitude of vector \(A\) times the magnitude of vector \(B\) times the cosine of the smaller angle \(\theta\) of the two angles between the two vectors. An equivalent definition of the dot product is given by

$$\mathbf{A} \cdot \mathbf{B} \equiv A_x B_x + A_y B_y + A_z B_z$$

(1.12)

The first definition could be considered a geometric definition of the dot product while the second definition could be considered an algebraic definition. With the use of the dot product, we can determine several useful quantities or properties associated with the combination of these two vectors. For instance, we can determine if two vectors are perpendicular or parallel to each other with the use of the dot product. In examining equation 1.11, we note that if \(A\) and \(B\) are perpendicular to each other, then the angle between them is 90°, and \(\cos (90°) = 0\), which means the dot product is equal to zero. In similar fashion, we note that if two vectors are parallel, then the magnitude of the dot product equals the product of the magnitudes of the two vectors. Finally, if we take the dot product of a vector with itself, we obtain the square of the magnitude of the vector, or
Another quantity we can obtain from the dot product is called the scalar projection of one vector onto another. For instance, if we want to obtain the scalar projection of the vector $\mathbf{A}$ onto the vector $\mathbf{B}$, we can compute this as follows:

$$\text{proj}_B \mathbf{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|}$$  \hspace{1cm} (1.14)

Note that this is a scalar quantity, and that we can also define the projection of the vector $\mathbf{B}$ onto the vector $\mathbf{A}$ in a similar fashion. To see the geometrical illustration of this, see Figure 1-3. We can obtain a more familiar form of the scalar projection by re-writing (1.14) using (1.11) to obtain

$$\text{proj}_B \mathbf{A} = |\mathbf{A}| \cos \theta$$  \hspace{1cm} (1.15)

Finally, we can simplify this even further if the vector $\mathbf{B}$ is a unit vector. In that case, the projection is simply the dot product:

$$\text{proj}_B \mathbf{A} = |\mathbf{A}| \cos \theta = \mathbf{A} \cdot \mathbf{a}_B$$  \hspace{1cm} (1.16)

One very important physical application of the scalar or dot product is the calculation of work. We can use the dot product to calculate the amount of work done when impressing a force on an object. For example, if we are to move an object a distance $\Delta x$ in the direction, $x$, we must apply a force, $\mathbf{F}$, with a component in the same direction. The total amount of work expended, $\Delta W$, is given by the expression

$$\Delta W = \mathbf{F} \cdot (\Delta x) \mathbf{a}_x = |\mathbf{F}| \Delta x \cos \theta$$  \hspace{1cm} (1.17)
This operation will be very useful later, when we start moving charges around in an electric field and we want to know how much work is required. We will also use the dot product to help us find the amount of flux crossing a surface. Other useful things that can be done using the dot product and its variations include finding the components of a vector if the other vector is a unit vector, or finding the direction cosines of a vector in three-dimensional space.

The scalar product obeys the commutative and distributive laws summarized as follows:

\[
\text{Commutative Law: } \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad (1.18)
\]

\[
\text{Distributive law: } \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad (1.19)
\]

The MATLAB command that permits taking a scalar product of the two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is either \( \text{dot}(\mathbf{A}, \mathbf{B}) \) or \( \text{dot} (\mathbf{B}, \mathbf{A}) \), since these are equal.

### 1.4.2 Vector (or Cross) Product

The second vector multiplication of two vectors is called the vector product, or the cross-product, and is defined as

\[
\mathbf{A} \times \mathbf{B} \equiv |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{a}_{\mathbf{A} \times \mathbf{B}} = A B \sin \theta \mathbf{a}_{\mathbf{A} \times \mathbf{B}} \quad (1.20)
\]

as illustrated in Figure 1-4.

**FIGURE 1–4**

Illustration of cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \)

This multiplication yields a vector whose direction is determined by the “right hand rule.” This rule states that if you take the fingers of your right hand (represented by vector \( \mathbf{A} \)) and curl them in the direction of vector \( \mathbf{B} \) to make a fist, the unit vector \( \mathbf{a}_{\mathbf{A} \times \mathbf{B}} \) will point in the direction of your thumb. Therefore, we find that the cross product is “anticommutative”: 
\[ \mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \quad (1.21) \]

or curling from vector \( \mathbf{B} \) to \( \mathbf{A} \) points the thumb in the opposite direction.

A convenient way to state that the two nonzero vectors are parallel (\( \theta = 0^\circ \)) or antiparallel (\( \theta = 180^\circ \)) is to use the vector product. If \( \mathbf{A} \times \mathbf{B} = 0 \), then the two vectors are parallel or antiparallel, since \( \sin 0^\circ = \sin 180^\circ = 0 \).

In Cartesian coordinates, we can easily calculate the vector product by remembering the expansion routine of the following determinant.

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix} = (\mathbf{A}_y \mathbf{B}_z - \mathbf{A}_z \mathbf{B}_y) \mathbf{a}_x + (\mathbf{A}_z \mathbf{B}_x - \mathbf{A}_x \mathbf{B}_z) \mathbf{a}_y + (\mathbf{A}_x \mathbf{B}_y - \mathbf{A}_y \mathbf{B}_x) \mathbf{a}_z
\quad (1.22)
\]

The MATLAB command that computes the vector product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is \texttt{cross(A, B)}. Remember \( \text{cross}(\mathbf{A}, \mathbf{B}) = -\text{cross}(\mathbf{B}, \mathbf{A}) \).

It is possible to give a geometric interpretation for the magnitude of the vector product. The magnitude \( |\mathbf{A} \times \mathbf{B}| \) is the area of the parallelogram whose sides are specified by the vectors \( \mathbf{A} \) and \( \mathbf{B} \) as shown in the Figure 1-5. From geometry, we recall that the area of a parallelogram with sides of length \( \mathbf{A} \) and \( \mathbf{B} \) with interior angle \( \theta \) is given by \( \text{Area} = \mathbf{A} \mathbf{B} \sin \theta \), which is also equal to the area of a rectangle with sides of length \( \mathbf{A} \) and \( \mathbf{B} \sin \theta \). By the definition of the cross product (1.22), this area is simply its magnitude: \( \text{Area} = |\mathbf{A} \times \mathbf{B}| \).

**FIGURE 1–5**
Parallelogram spanned by vectors by \( \mathbf{A} \) and \( \mathbf{B} \)

### 1.4.3 Triple Products

Two triple products encountered in electromagnetic theory are included here. The first is called the **scalar triple product**. It is defined, following the cyclical permutation, as
\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \]  

(1.23)

It can also be written as a 3 by 3 determinant:

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix}
A_x & A_y & A_z \\
B_x & B_y & B_z \\
C_x & C_y & C_z
\end{vmatrix}
\]

(1.24)

In the following, we show that the volume of a parallelepiped defined by three vectors originating at a point can be defined in terms of the scalar and vector products of the vectors. As illustrated in Figure 1-6, the volume of the parallelepiped is given by

\[ \text{Volume} = (\text{area of the base of the parallelepiped}) \times (\text{height of the parallelepiped}) \\
= (|\mathbf{A} \times \mathbf{B}|) (|\mathbf{C} \cdot \mathbf{a}_n|) = (|\mathbf{A} \times \mathbf{B}|) \left| \mathbf{C} \cdot \left( \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} \right) \right| = |\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})| \]  

(1.25)

Note that the height of the parallelepiped is the projection of vector \( \mathbf{C} \) onto the unit vector \( (\mathbf{A} \times \mathbf{B})/|\mathbf{A} \times \mathbf{B}| \) that is perpendicular to the base.

\[ \text{FIGURE 1–6} \]
Parallelepiped spanned by three vectors \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \).

The second triple product is called the vector triple product, such as \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \). It can be shown that

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \]  

(1.26)

This triple product is sometimes called the “bac-cab” rule, since this is an easy way to remember how the vectors are ordered. The inclusion of the parentheses in this vector triple product is critical since it does not, in general, obey the associative law, that is
EXAMPLE 1.2

Given three vectors \( \mathbf{A} = -a_x + 2a_y + 3a_z \), \( \mathbf{B} = 3a_x + 4a_y + 5a_z \) and \( \mathbf{C} = 2a_x - 2a_y + 7a_z \), compute
(a) the scalar product \( \mathbf{A} \cdot \mathbf{B} \)
(b) the angle between \( \mathbf{A} \) and \( \mathbf{B} \)
(c) the scalar projection of \( \mathbf{A} \) on \( \mathbf{B} \)
(d) the vector product \( \mathbf{A} \times \mathbf{B} \)
(e) the area of the parallelogram whose sides are specified by \( \mathbf{A} \) and \( \mathbf{B} \)
(f) the volume of a parallelepiped defined by vectors \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \)
(g) the vector triple product \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) and check equation (1.26)

Solution:

(a) The scalar product \( \mathbf{A} \cdot \mathbf{B} \) is given by
\[
\mathbf{A} \cdot \mathbf{B} = -1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 20.
\]
(b) The angle between the two vectors is computed from the definition of the scalar product.
\[
\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{||\mathbf{A}|| ||\mathbf{B}||} = \frac{20}{\sqrt{(-1)^2 + 2^2 + 3^2} \sqrt{3^2 + 4^2 + 5^2}} = 0.7559 \text{ or } \theta = 40.89^\circ
\]
(c) \( \text{proj}_B \mathbf{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{||\mathbf{B}||^2} ||\mathbf{B}|| = \frac{20}{\sqrt{3^2 + 4^2 + 5^2}} = 2.8284 \)

(d) The vector product \( \mathbf{A} \times \mathbf{B} \) is given by
\[
\begin{vmatrix}
a_x & a_y & a_z \\
-1 & 2 & 3 \\
3 & 4 & 5 \\
\end{vmatrix} = -2a_x + 14a_y - 10a_z
\]

(e) The area = \( ||\mathbf{A} \times \mathbf{B}|| = \sqrt{(-2)^2 + 14^2 + (-10)^2} = 17.32 \)

(f) The scalar triple product \( \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = -2 \cdot 2 + 14 \cdot (-2) - 10 \cdot 7 = -102 \)

The volume of a parallelepiped defined by vectors \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) is \( \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = -102 \)

(g) The vector product \( \mathbf{B} \times \mathbf{C} \) is given by
\[
\begin{vmatrix}
a_x & a_y & a_z \\
3 & 4 & 5 \\
2 & -2 & 7 \\
\end{vmatrix} = 38a_x - 11a_y - 14a_z
\]

Then we have
\[
\begin{vmatrix}
a_x & a_y & a_z \\
-1 & 2 & 3 \\
38 & -11 & -14 \\
\end{vmatrix} = 5a_x + 100a_y - 65a_z
\]

The scalar product \( \mathbf{A} \cdot \mathbf{C} = -1 \cdot 1 + 2 \cdot (-2) + 3 \cdot 7 = 15 \), then
\[
\mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) = 15\mathbf{B} - 20\mathbf{C} = 15(3a_x + 4a_y + 5a_z) - 20(2a_x - 2a_y + 7a_z)
\]
\[
= 5a_x + 100a_y - 65a_z
\]
which is the same as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

**MATLAB Solution:**

The following MATLAB source code can be used to solve the problem and get the same answer as shown in the solution above.

```matlab
A = [-1 2 3];
B = [3 4 5];
C = [2 -2 7];
S = dot(A,B)
theta = acos(S/norm(A)/norm(B)) * 180/pi  % in degrees
a_B = B/norm(B)
projAontoB = dot(A,a_B)
T = cross(A,B)
areaAB = norm(T)
volABC = abs(dot(C,T))
Q = cross(B,C);
leftside = cross(A,Q)
rightside = B*dot(A,C) - C*dot(A,B)
```

These source codes are in ex102.m. In the m-file, we have also added plotting functions. Readers can read the details of the file and run it.

**MATLAB figure for EXAMPLE 1.2**

1.5  Coordinate Systems
In this text, we will frequently encounter problems where there is a source of an electromagnetic field. To be able to specify the field at a point in space caused by a source, we have to refer to a coordinate system. In three dimensions, the coordinate system can be specified by the intersection of three surfaces. An orthogonal coordinate system is defined when these three surfaces are mutually orthogonal at every point. Coordinate surfaces may be planar or curved. A general orthogonal coordinate system is illustrated in Figure 1–7.

**Figure 1–7**
A general orthogonal coordinate system. Three surfaces intersect at a point, and the unit vectors are mutually orthogonal at that point.

In Cartesian coordinates, all of the three surfaces are planes, and they are specified by each of the independent variables \( x, y, \) and \( z \) separately having prescribed values. In cylindrical coordinates, the surfaces are two planes and a cylinder. In spherical coordinates, the surfaces are a sphere, a plane, and a cone. We will examine each of these in detail in the following discussion. There are many other coordinate systems that can be employed for particular problems, and there are formulas that allow one to easily transform vectors from one system to another.

The three coordinate systems used in this text are pictured in Figure 1–8 as (a), (b), and (c). The directions along the axes of the coordinate systems are given by the sets of unit vectors \( (\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z) \), \( (\mathbf{a}_r, \mathbf{a}_\phi, \mathbf{a}_z) \), and \( (\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi) \) for Cartesian, cylindrical, and spherical coordinates, respectively. In each of the coordinate systems, the unit vectors are mutually orthogonal at every point.
In each coordinate system, the unit vectors point in the direction of increasing coordinate value. In Cartesian coordinates, the direction of the unit vectors is independent of position, whereas in cylindrical and spherical coordinates, unit vector directions at a point in space depend on the location of that point. For example, in spherical coordinates, the unit vector \( \mathbf{a}_r \) is directed radially away from the origin at every point in space; it will be directed in the \(+z\) direction if \( \theta = 0 \), and it will be directed in the \(-z\) direction if \( \theta = \pi \). Since we will employ these three coordinate systems extensively in the following chapters, it is useful to summarize the important properties of each one.

### 1.2.1 Cartesian Coordinates

The unit vectors in Cartesian coordinates depicted in Figure 1–8a are normal to the intersection of three planes as shown in Figure 1–9. Each of the surfaces depicted in this figure is a plane that is individually normal to a coordinate axis.

![Figure 1–8](image1)

**FIGURE 1–8**
The three coordinate systems that will be employed in this text. The unit vectors are indicated. (a) Cartesian coordinates. (b) Cylindrical coordinates. (c) Spherical coordinates.

![Figure 1–9](image2)

**FIGURE 1–9**
A point in Cartesian coordinates is defined by the intersection of the three planes: \( x = \) constant; \( y = \) constant; \( z = \) constant. The three unit vectors are normal to each of the three surfaces.
For the unit vectors that are in the directions of the $x$, $y$, and $z$ axes, we can easily prove that

\[
\begin{align*}
\mathbf{a}_x \cdot \mathbf{a}_x &= \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \\
\mathbf{a}_x \cdot \mathbf{a}_y &= \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_y \cdot \mathbf{a}_z = 0
\end{align*}
\]  

(1.28)

The following rules also apply to the cross products of the unit vectors, since this is a right-handed system:

\[
\begin{align*}
\mathbf{a}_x \times \mathbf{a}_y &= \mathbf{a}_z \\
\mathbf{a}_y \times \mathbf{a}_z &= \mathbf{a}_x \\
\mathbf{a}_z \times \mathbf{a}_x &= \mathbf{a}_y
\end{align*}
\]  

(1.29)

All other cross products of unit vectors follow from the facts that the cross product is anti-symmetric ($\mathbf{a}_x \times \mathbf{a}_y = -\mathbf{a}_y \times \mathbf{a}_x$, etc.), and the cross product of any vector with itself is zero ($\mathbf{a}_x \times \mathbf{a}_x = 0$, etc.).

In Figure 1–10, the position vector $\mathbf{r}_P$ (or $\mathbf{P}$) from the origin to a point $P(x_P, y_P, z_P)$ in Cartesian coordinates is defined as

\[
\mathbf{r}_P (\text{or } \mathbf{P}) = x_P \mathbf{a}_x + y_P \mathbf{a}_y + z_P \mathbf{a}_z
\]  

(1.30)

and the distance vector that extends from point $P$ to point $Q(x_Q, y_Q, z_Q)$ is

\[
\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P \text{ (or } \mathbf{Q} - \mathbf{P})
\]

\[
= (x_Q - x_P) \mathbf{a}_x + (y_Q - y_P) \mathbf{a}_y + (z_Q - z_P) \mathbf{a}_z
\]  

(1.31)

**FIGURE 1–10**
Illustration of position vector
EXAMPLE 1.3

There are four points $A(1,2,3)$, $B(4,5,4)$, $C(3,-3,8)$ and $D(2,3,7)$ in Cartesian coordinate system. Find

1. $R_{AB}$, $R_{AC}$ and $R_{AD}$
2. the area of triangle ABC
3. the volume of tetrahedral ABCD

Solution:

1. $R_{AB} = (4-1)a_x + (5-2)a_y + (4-3)a_z = 3a_x + 3a_y + a_z$
   $R_{AC} = (3-1)a_x + (-3-2)a_y + (8-3)a_z = 2a_x - 5a_y + 5a_z$
   $R_{AD} = (2-1)a_x + (3-2)a_y + (7-3)a_z = a_x + a_y + 4a_z$

2. The area of the triangle ABC is equal to half of the area of the parallelogram spanned by $R_{AB}$ and $R_{AC}$. We calculate
   \[
   \begin{vmatrix}
   a_x & a_y & a_z \\
   3 & 3 & 1 \\
   2 & -5 & 5 
   \end{vmatrix}
   = 20a_x - 13a_y - 21a_z
   \]
   Then, the area of triangle ABC
   \[
   = \frac{1}{2} |R_{AB} \times R_{AC}| = \frac{1}{2} \sqrt{20^2 + (-13)^2 + (-21)^2} = 15.8902
   \]

3. The volume of the tetrahedral ABCD is equal to $1/6$ of the volume of the parallelepiped defined by $R_{AB}$, $R_{AC}$ and $R_{AD}$. We calculate
   \[
   R_{AD} \times (R_{AB} \times R_{AC}) = 1 \times 20 + 1 \times (-13) + 4 \times (-21) = -77
   \]
   Therefore the volume of tetrahedral ABCD
   \[
   \frac{1}{6} |R_{AD} \times (R_{AB} \times R_{AC})| = 12.8333
   \]

MATLAB Solution:

The following MATLAB source code can be used to get the same answer as shown in above solution.

```matlab
A = [1 2 3];
B = [4 5 4];
C = [3 -3 8];
D = [2 3 7];
R_AB = B - A;
R_AC = C - A;
R_AD = D - A;
T = cross(R_AB,R_AC);
Area_ABC = norm(T)/2
Volume_ABCD = abs(dot(R_AD,T))/6
```

The above source code is included in ex103.m. The m file also plots triangle ABC and tetrahedral ABCD.
We can define a time-varying vector field \( \mathbf{F}(x,y,z,t) \) whose three components are functions of position \((x,y,z)\) and time \(t\) in Cartesian coordinate system as

\[
\mathbf{F}(x,y,z,t) = F_x(x,y,z,t)\mathbf{a}_x + F_y(x,y,z,t)\mathbf{a}_y + F_z(x,y,z,t)\mathbf{a}_z \quad (1.32)
\]

If a vector field \( \mathbf{G}(x,y,z) \) is static or time-invariant, we have

\[
\mathbf{G}(x,y,z) = G_x(x,y,z)\mathbf{a}_x + G_y(x,y,z)\mathbf{a}_y + G_z(x,y,z)\mathbf{a}_z \quad (1.33)
\]

**EXAMPLE 1.4**

A vector field \( \mathbf{A} \) in two dimensional space is given as \( \mathbf{A}(x,y) = 4x^2\mathbf{a}_x + 2xy\mathbf{a}_y \). Find

1. the unit vectors of \( \mathbf{A} \) at \((1, -1)\) and \((-2, 3)\)
2. plot \( A_x \) versus \( x \) for \(-1 \leq x \leq 1\) using MATLAB
3. plot \( A_y \) versus \( x \) and \( y \) for \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\) using MATLAB function \texttt{surf}
4. plot \( \mathbf{A} \) using MATLAB function \texttt{quiver} for \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\)

**Solution:**

1. We can calculate the values of vector field \( \mathbf{A} \) at \((1, -1)\) and \((-2, 3)\) as follows

\[
\mathbf{A}(1,-1) = 4 \times 1^2 \mathbf{a}_x + 2 \times 1 \times (-1) \mathbf{a}_y = 4\mathbf{a}_x - 2\mathbf{a}_y
\]

\[
\mathbf{A}(-2,3) = 4 \times (-2)^2 \mathbf{a}_x + 2 \times (-2) \times 3 \mathbf{a}_y = 16\mathbf{a}_x - 12\mathbf{a}_y
\]

Then, the unit vectors at these two points are

\[
\mathbf{a}_{\mathbf{A}(1,-1)} = \frac{4\mathbf{a}_x - 2\mathbf{a}_y}{\sqrt{4^2 + (-2)^2}} = 0.8944\mathbf{a}_x - 0.4472\mathbf{a}_y
\]

\[
\mathbf{a}_{\mathbf{A}(-2,3)} = \frac{16\mathbf{a}_x - 12\mathbf{a}_y}{\sqrt{16^2 + (-12)^2}} = 0.8\mathbf{a}_x - 0.6\mathbf{a}_y
\]
(2) and (4) are only solved using MATLAB.

MATLAB Solution:

(1) We can use MATLAB symbolic operations to express a vector field. The symbolic operation is easy for students to understand and the student version of MATLAB has the symbolic toolbox. Firstly, we define $x$, $y$ and $z$ as symbolic variables using MATLAB command syms as

\[ \text{syms } x \ y \]

And then we can write down vector field $\mathbf{A}$ as

\[ \mathbf{A} = [4x^2, \ 2xy] \]

For the values of $\mathbf{A}$ at specific points, we can use MATLAB command subs to obtain.

\[ \mathbf{A}_{\text{point1}} = \text{subs}(\mathbf{A}, \{x,y\}, \{1,-1\}) \]
\[ \mathbf{A}_{\text{point2}} = \text{subs}(\mathbf{A}, \{x,y\}, \{-2,3\}) \]

And the unit vectors can be obtained as

\[ \mathbf{a}_{\mathbf{A}1} = \frac{\mathbf{A}_{\text{point1}}}{\|\mathbf{A}_{\text{point1}}\|} \]
\[ \mathbf{a}_{\mathbf{A}2} = \frac{\mathbf{A}_{\text{point2}}}{\|\mathbf{A}_{\text{point2}}\|} \]

(2) We can get the x component of $\mathbf{A}$ from

\[ \mathbf{Ax} = \mathbf{A}(1); \]

To plot $\mathbf{Ax}$ using MATLAB function plot for $x$ from $-2$ to $2$, we need to calculate numerical values of $\mathbf{Ax}$ as follows

\[ \mathbf{xx} = -1:0.1:1; \]
\[ \mathbf{Axx} = \text{subs}(\mathbf{Ax}, \{x\}, \{\mathbf{xx}\}); \]

And then, we can simply plot as follows

\[ \text{plot}(\mathbf{xx}, \mathbf{Axx}); \]

(3) We can get $\mathbf{Ay}$ from

\[ \mathbf{Ay} = \mathbf{A}(2); \]

To plot using surf, we need to build a mesh using MATLAB function meshgrid.

\[ [\mathbf{X}, \mathbf{Y}] = \text{meshgrid}(-1:0.1:1, \ -1:0.1:1) \]

And then, we calculate numerical values of $\mathbf{Ay}$ on this mesh using subs

\[ \mathbf{Ay}_{\text{num}} = \text{subs}(\mathbf{Ay}, \{x,y\}, \{\mathbf{X}, \mathbf{Y}\}) \]

After that, we can plot $\mathbf{Ay}$ using 3D MATLAB plot function surf

\[ \text{surf}(\mathbf{X}, \mathbf{Y}, \mathbf{Ay}_{\text{num}}) \]

(4) We can also calculate $\mathbf{Ax}$ on the mesh although it only depends on $x$. That is,

\[ \mathbf{Ax}_{\text{num}} = \text{subs}(\mathbf{Ax}, \{x,y\}, \{\mathbf{X}, \mathbf{Y}\}) \]

And then, the vector field $\mathbf{A}(x,y)$ can be plotted using quiver.

\[ \text{quiver}(\mathbf{X}, \mathbf{Y}, \mathbf{Ax}_{\text{num}}, \mathbf{Ay}_{\text{num}}) \]

In quiver plot, the magnitude and direction of the vector field at any point are indicated by the length and orientation of the arrows. In all the figures plotted,
we can add labels for all the axes and title for each figure. These details were included in the MATLAB source code ex104.m.

MATLAB Figure for Example 1.4 (2)

MATLAB Figure for Example 1.4 (3)
We will perform line, surface, and volume integrals in the following chapters. Figure 1–11 depicts the differential line element, surface elements, and volume elements in Cartesian coordinates. The differential length vector $\mathbf{dl}$ is defined as

$$\mathbf{dl} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$$

(1.34)

where $dx$, $dy$ and $dz$ are differential lengths along $ax$, $ay$ and $az$ directions respectively.

Note that there are six possible differential surface elements, each corresponding to one of the six faces of the differential volume. In each case, the vector direction is the outward normal direction. The differential surface areas normal to $ax$, $ay$ and $az$ directions are

$$\begin{align*}
    ds_x &= dydz\mathbf{a}_x \\
    ds_y &= dxdz\mathbf{a}_y \\
    ds_z &= dxdy\mathbf{a}_z
\end{align*}$$

(1.35)

The differential volume element $dv$ in Cartesian coordinate system is defined as the product of the three differential lengths. That is,

$$dv = dxdydz$$

(1.36)
FIGURE 1–11
In Cartesian coordinates, a differential line element \( dl = dx \mathbf{a}_x + dy \mathbf{a}_y + dz \mathbf{a}_z \) is shown. Three of six differential surface elements, \( ds_x = dydz \mathbf{a}_x \), \( ds_y = dxdz \mathbf{a}_y \), and \( ds_z = dxdy \mathbf{a}_z \) are shown along with the differential volume element \( dv = dxdydz \).

1.2.2 Cylindrical Coordinates

The unit vectors in cylindrical coordinates depicted in Figure 1–8b are normal to the intersection of three surfaces as shown in Figure 1–12. Two of the surfaces depicted in this figure are planes, and the third surface is a cylinder that is centered on the \( z \) axis. A point \((\rho, \phi, z)\) in cylindrical coordinates is located at the intersection of the two planes and the cylinder. The value of \( \rho \) is the distance away from the \( z \) axis and the value of \( \phi \) is the angle between the projection onto the \( x - y \) plane and the \( x \) axis. The mutually-perpendicular unit vectors \( \mathbf{a}_\rho \), \( \mathbf{a}_\phi \), and \( \mathbf{a}_z \) are in the direction of increasing coordinate value; note that unlike Cartesian unit vectors, the directions of \( \mathbf{a}_\rho \) and \( \mathbf{a}_\phi \) vary with location.
FIGURE 1–12
The cylindrical coordinate system. A point is located at the intersection of a cylinder and two planes. The variables $\rho$, $\phi$, and $z$ are shown. A differential line element $d\ell$, differential surface elements $d\ell_{\rho}$, $d\ell_{\phi}$, and $d\ell_{z}$, and a differential volume element $dV$ are depicted.
As usual, the dot product of a unit vector with itself is equal to one, and the dot product of one unit vector with another is equal to zero. That is
\[
\begin{align*}
\mathbf{a}_\rho \cdot \mathbf{a}_\rho &= \mathbf{a}_\phi \cdot \mathbf{a}_\phi = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \\
\mathbf{a}_\rho \cdot \mathbf{a}_\phi &= \mathbf{a}_\rho \cdot \mathbf{a}_z = \mathbf{a}_\phi \cdot \mathbf{a}_z = 0
\end{align*}
\] (1.37)
Also, since this is a right-handed system, the cross products are given by
\[
\begin{align*}
\mathbf{a}_\rho \times \mathbf{a}_\phi &= \mathbf{a}_z \\
\mathbf{a}_\phi \times \mathbf{a}_z &= \mathbf{a}_\rho \\
\mathbf{a}_z \times \mathbf{a}_\rho &= \mathbf{a}_\phi
\end{align*}
\] (1.38)
The negative of these results holds when the terms are interchanged, and the cross product of any unit vector with itself is zero.

The differential line element \( d\mathbf{l} \) in cylindrical coordinates can be expressed as
\[
\begin{align*}
d\mathbf{l} &= d\rho \mathbf{a}_\rho + \rho d\phi \mathbf{a}_\phi + dz \mathbf{a}_z
\end{align*}
\] (1.39)
where \( d\rho \), \( \rho d\phi \), and \( dz \) are differential lengths along \( \mathbf{a}_\rho \), \( \mathbf{a}_\phi \), and \( \mathbf{a}_z \) directions, respectively. The differential surface elements \( ds_\rho \), \( ds_\phi \), and \( ds_z \) which are perpendicular to \( \mathbf{a}_\rho \), \( \mathbf{a}_\phi \), and \( \mathbf{a}_z \) directions, respectively, are given as
\[
\begin{align*}
ds_\rho &= \rho d\phi dz \mathbf{a}_\rho \\
ds_\phi &= d\rho dz \mathbf{a}_\phi \\
ds_z &= \rho d\rho d\phi \mathbf{a}_z
\end{align*}
\] (1.40)
Finally, the differential volume element \( dv \) is given as the product of differential lengths as
\[
dv = \rho d\rho d\phi dz
\] (1.41)
The coordinates of any point can be transformed from spherical coordinates \( (r, \theta, \phi) \) to Cartesian coordinates \( (x,y,z) \). Figure 1–13 shows the relationship between Cartesian and cylindrical coordinates.
FIGURE 1–13
Relationship between Cartesian and cylindrical coordinates.

From Figure 1–13, the transformation is found to be

\[
\begin{align*}
\begin{cases}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{cases}
\end{align*}
\] (1.42)

The inverse transformation from Cartesian to spherical coordinates is
\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2} \\
\phi &= \tan^{-1}\left(\frac{y}{x}\right) \\
z &= z
\end{align*}
\] (1.43)

Care must be used in choosing the correct quadrant for the arc-tangent function in \(\phi\) by considering the signs of \(x\) and \(y\).

Using (1.42) and (1.43), the scalar field in one coordinate system can be easily transformed into another coordinate system.

A vector field \(\mathbf{A}\) can be expressed in cylindrical and Cartesian coordinates as

\[
\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z
\] (1.44)

The relationship between vector field components in the two coordinate systems may be found using the scalar product and recalling that the scalar product of a vector with a unit vector may be interpreted as the amount of the vector in the direction of the unit vector. Note that \(A_z\) is identical in both coordinate systems. Now, the \(x\) and \(y\) components of a vector may be found as the dot product of the vector with the unit vector \(\mathbf{a}_x\) and \(\mathbf{a}_y\), respectively. Given \(\mathbf{A}\) in cylindrical coordinates, this means

\[
\begin{align*}
A_x &= \mathbf{A} \cdot \mathbf{a}_x = (A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z) \cdot \mathbf{a}_x = A_\rho \mathbf{a}_\rho \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x \\
A_y &= \mathbf{A} \cdot \mathbf{a}_y = (A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z) \cdot \mathbf{a}_y = A_\rho \mathbf{a}_\rho \cdot \mathbf{a}_y + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_y
\end{align*}
\] (1.45)

Equation (1.45) together with \(A_z\) can be written in matrix form as

\[
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix} =
\begin{bmatrix}
\mathbf{a}_x \cdot \mathbf{a}_x & \mathbf{a}_x \cdot \mathbf{a}_\rho & 0 \\
\mathbf{a}_y \cdot \mathbf{a}_x & \mathbf{a}_y \cdot \mathbf{a}_\rho & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A_\rho \\
A_\phi \\
A_z
\end{bmatrix}
\] (1.46)

From Figure 1–13, we can find these dot product in (1.46). The results are given in Table 1–1.

**Table 1–1** Dot products of unit vectors in Cartesian and cylindrical coordinate systems

<table>
<thead>
<tr>
<th></th>
<th>(\mathbf{a}_x)</th>
<th>(\mathbf{a}_\rho)</th>
<th>(\mathbf{a}_z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbf{a}_x \cdot)</td>
<td>(\cos \phi)</td>
<td>(-\sin \phi)</td>
<td>0</td>
</tr>
<tr>
<td>(\mathbf{a}_y \cdot)</td>
<td>(\sin \phi)</td>
<td>(\cos \phi)</td>
<td>0</td>
</tr>
<tr>
<td>(\mathbf{a}_z \cdot)</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore (1.46) becomes

\[
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A_\rho \\
A_\phi \\
A_z
\end{bmatrix}
\] (1.47)

or
\[
\begin{align*}
A_x &= A_\rho \cos \phi - A_\theta \sin \phi \\
A_y &= A_\rho \cos \phi + A_\theta \cos \phi \\
A_z &= A_z
\end{align*}
\]
(1.48)

where \( \sin \phi \) and \( \cos \phi \) can be expressed in Cartesian coordinates as

\[
\begin{align*}
\sin \phi &= \frac{y}{\sqrt{x^2 + y^2}} \\
\cos \phi &= \frac{x}{\sqrt{x^2 + y^2}}
\end{align*}
\]
(1.49)

Likewise, we can also obtain vector field components in cylindrical coordinates from Cartesian coordinates as

\[
\begin{bmatrix}
A_\rho \\
A_\theta \\
A_z
\end{bmatrix} =
\begin{bmatrix}
0 & A_x \\
0 & A_y \\
1 & A_z
\end{bmatrix}
\begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix}
\]
(1.50)

or

\[
\begin{align*}
A_\rho &= A_z \cos \phi + A_\theta \sin \phi \\
A_\theta &= -A_z \sin \phi + A_\theta \cos \phi \\
A_z &= A_z
\end{align*}
\]
(1.51)

1.2.3 Spherical Coordinates

The unit vectors in spherical coordinates depicted in Figure 1–8c are normal to the intersection of three surfaces as shown in Figure 1–14. One of the surfaces depicted in this figure is a plane, another surface is a sphere, and the third surface is a cone. The latter two surfaces are centered on the \( z \) axis. A point in spherical coordinates is specified by the intersection of the three surfaces. The unit vectors \( a_r, a_\theta \) and \( a_\phi \) are perpendicular to the sphere, the cone, and the plane. The variables and unit vectors in spherical coordinates are also shown in the figure.
FIGURE 1–14
Spherical coordinates. A point is defined by the intersection of a sphere whose radius is \( r \), a plane that makes an angle \( \phi \) with respect to the \( x \) axis, and a cone that makes an angle \( \theta \) with respect to the \( z \) axis. A differential line element \( d\ell \), differential surface elements \( ds_\phi \), \( ds_\theta \), and \( ds_z \), and a differential volume element \( dV \) are depicted.

A point \((r, \theta, \phi)\) in spherical coordinates is located at the intersection of the sphere, cone and plane. The value of \( r \) is the distance away from the origin, \( \theta \) is the angle from the \( z \) axis, and \( \phi \) is the same angle as in cylindrical coordinates. The mutually-
perpendicular unit vectors \( \mathbf{a}_r, \mathbf{a}_\theta \) and \( \mathbf{a}_\phi \) are in the direction of increasing coordinate value; note that unlike Cartesian unit vectors, the directions of the unit vectors vary with location.

As usual, the dot product of a unit vector with itself is equal to one, and the dot product of one unit vector with another is equal to zero. That is

\[
\begin{align*}
\mathbf{a}_r \cdot \mathbf{a}_r &= \mathbf{a}_\theta \cdot \mathbf{a}_\theta = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = 1 \\
\mathbf{a}_r \cdot \mathbf{a}_\theta &= \mathbf{a}_r \cdot \mathbf{a}_\phi = \mathbf{a}_\theta \cdot \mathbf{a}_\phi = 0
\end{align*}
\]

(1.52)

Also, since this is a right-handed system, the cross products are given by

\[
\begin{align*}
\mathbf{a}_r \times \mathbf{a}_\theta &= \mathbf{a}_\phi \\
\mathbf{a}_\theta \times \mathbf{a}_\phi &= \mathbf{a}_r \\
\mathbf{a}_\phi \times \mathbf{a}_r &= \mathbf{a}_\theta
\end{align*}
\]

(1.53)

The negative of these results holds when the terms are interchanged, and the cross product of any unit vector with itself is zero.

The differential line element \( dl \) in spherical coordinates can be expressed as

\[
dl = dr \mathbf{a}_r + rd\theta \mathbf{a}_\theta + r \sin \theta d\phi \mathbf{a}_\phi
\]

(1.54)

where \( dr, rd\theta \) and \( r \sin \theta d\phi \) are differential lengths along \( \mathbf{a}_r \), \( \mathbf{a}_\theta \), and \( \mathbf{a}_\phi \) directions, respectively. The differential surface elements \( ds_r, ds_\theta \) and \( ds_\phi \) which are perpendicular to \( \mathbf{a}_r, \mathbf{a}_\theta \) and \( \mathbf{a}_\phi \) directions, respectively, are given as

\[
\begin{align*}
(ds_r = (rd\theta)(r \sin \theta d\phi) \mathbf{a}_r &= r^2 \sin \theta d\theta d\phi \mathbf{a}_r \\
(ds_\theta = dr(r \sin \theta d\phi) \mathbf{a}_\theta &= r \sin \theta dr d\phi \mathbf{a}_\theta \\
(ds_\phi = dr(rd\theta) \mathbf{a}_\phi &= r \sin \theta dr d\theta \mathbf{a}_\phi
\end{align*}
\]

(1.55)

Finally, the differential volume element \( dv \) is given as the product of differential lengths as

\[
dv = dr(rd\theta)(r \sin \theta d\phi) = r^2 \sin \theta dr d\theta d\phi
\]

(1.56)

In the following, we are going to look at the transformation between spherical and cylindrical coordinates and the transformation between spherical and Cartesian coordinates.

\textbf{Figure 1–15} shows the relationship between spherical and cylindrical coordinates. From this figure, we can find the transformation of the coordinate variables from spherical to cylindrical coordinates is

\[
\begin{align*}
\rho &= r \sin \theta \\
\phi &= \phi \\
z &= r \cos \theta
\end{align*}
\]

(1.57)

and the transformation of the coordinate variables from cylindrical to spherical coordinates yields
\[
\begin{align*}
    r &= \sqrt{\rho^2 + z^2} \\
    \theta &= \cos^{-1}\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right) \\
    \phi &= \phi
\end{align*}
\] (1.58)

**FIGURE 1–15**
Relationship between cylindrical and spherical coordinates.
A vector field \( \mathbf{A} \) can be expressed in cylindrical and spherical coordinates as

\[
\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi
\]

The relationship between vector field components in the two coordinate systems may be found using the dot product and recalling that the dot product of a vector with a unit vector is the projection of the vector onto the direction of the unit vector. Note that \( A_\phi \) is identical in both coordinate systems. Now, the \( \rho \) and \( z \) components of a vector may be found as the dot product of the vector with the unit vector \( \mathbf{a}_\rho \) and \( \mathbf{a}_z \), respectively. Given \( \mathbf{A} \) in cylindrical coordinates, this means

\[
\begin{bmatrix}
A_\rho \\
A_\theta \\
A_\phi
\end{bmatrix} =
\begin{bmatrix}
\mathbf{a}_r \cdot \mathbf{a}_\rho & \mathbf{a}_\theta \cdot \mathbf{a}_\rho & 0 \\
0 & 0 & 1 \\
\mathbf{a}_z \cdot \mathbf{a}_\rho & \mathbf{a}_z \cdot \mathbf{a}_\theta & 0
\end{bmatrix}
\begin{bmatrix}
A_r \\
A_\theta \\
A_\phi
\end{bmatrix}
\]

Equation (1.45) together with \( A_\phi \) can be written in matrix form as

\[
\begin{bmatrix}
A_\rho \\
A_\theta \\
A_\phi
\end{bmatrix} =
\begin{bmatrix}
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\cos \theta & -\sin \theta & 0
\end{bmatrix}
\begin{bmatrix}
A_r \\
A_\theta \\
A_\phi
\end{bmatrix}
\]

From Figure 1–15, we can find these dot products in (1.61). The results are given in Table 1–2.

**TABLE 1–2** Dot products of unit vectors in cylindrical and spherical coordinate systems

<table>
<thead>
<tr>
<th>( \mathbf{a}_\rho \cdot )</th>
<th>( \mathbf{a}_\theta \cdot )</th>
<th>( \mathbf{a}_\phi \cdot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{a}_r )</td>
<td>( \sin \theta )</td>
<td>( \cos \theta )</td>
</tr>
<tr>
<td>( \mathbf{a}_\theta )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbf{a}_z )</td>
<td>( \cos \theta )</td>
<td>( -\sin \theta )</td>
</tr>
</tbody>
</table>

Therefore (1.61) becomes

\[
\begin{bmatrix}
A_\rho \\
A_\theta \\
A_\phi
\end{bmatrix} =
\begin{bmatrix}
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\cos \theta & -\sin \theta & 0
\end{bmatrix}
\begin{bmatrix}
A_r \\
A_\theta \\
A_\phi
\end{bmatrix}
\]

or

\[
\begin{align*}
A_\rho &= A_r \sin \theta + A_\theta \cos \theta \\
A_\theta &= A_\phi \\
A_\phi &= A_r \cos \theta - A_\theta \sin \theta
\end{align*}
\]

where \( \sin \theta \) and \( \cos \theta \) can be expressed in cylindrical coordinates as
Likewise, we can also obtain vector field components in cylindrical coordinates from Cartesian coordinates as

\[
\begin{bmatrix}
A_r \\
A_\theta \\
A_\phi
\end{bmatrix} =
\begin{bmatrix}
a_r \cdot \mathbf{a}_r & a_r \cdot \mathbf{a}_\theta & a_r \cdot \mathbf{a}_\phi \\
a_\theta \cdot \mathbf{a}_r & a_\theta \cdot \mathbf{a}_\theta & a_\theta \cdot \mathbf{a}_\phi \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
A_\rho \\
A_\phi \\
A_\theta
\end{bmatrix} =
\begin{bmatrix}
\sin \theta & 0 & \cos \theta \\
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
A_r \\
A_\phi \\
A_\theta
\end{bmatrix}
\] (1.65)

or

\[
\begin{align*}
A_r &= A_\rho \sin \theta + A_\phi \cos \theta \\
A_\phi &= A_\rho \cos \theta - A_\phi \sin \theta \\
A_\theta &= A_z
\end{align*}
\] (1.66)

From (1.42) and (1.57), we find the transformation of the coordinate variables from spherical to Cartesian coordinates is

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\] (1.67)

and from (1.43) and (1.58), the transformation of the coordinate variables from Cartesian to spherical coordinates yields

\[
\begin{align*}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\
\phi &= \tan^{-1}\left(\frac{y}{x}\right)
\end{align*}
\] (1.68)

Once again, attention must be paid to the quadrant of the result of the arctangent.

A vector field \( \mathbf{A} \) can be expressed in spherical and Cartesian coordinates as

\[
\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi
\] (1.69)

Now, the \( x, y, \) and \( z \) components of a vector may be found as the dot product of the vector with the unit vector \( \mathbf{a}_x, \mathbf{a}_y, \) and \( \mathbf{a}_z \), respectively. Given \( \mathbf{A} \) in spherical coordinates, this means

\[
\begin{align*}
A_x &= \mathbf{A} \cdot \mathbf{a}_x = \mathbf{a}_x \cdot (A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi) = \mathbf{a}_x \cdot A_r \mathbf{a}_r + \mathbf{a}_x \cdot A_\theta \mathbf{a}_\theta + \mathbf{a}_x \cdot A_\phi \mathbf{a}_\phi \\
A_y &= \mathbf{A} \cdot \mathbf{a}_y = \mathbf{a}_y \cdot (A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi) = \mathbf{a}_y \cdot A_r \mathbf{a}_r + \mathbf{a}_y \cdot A_\theta \mathbf{a}_\theta + \mathbf{a}_y \cdot A_\phi \mathbf{a}_\phi \\
A_z &= \mathbf{A} \cdot \mathbf{a}_z = \mathbf{a}_z \cdot (A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi) = \mathbf{a}_z \cdot A_r \mathbf{a}_r + \mathbf{a}_z \cdot A_\theta \mathbf{a}_\theta + \mathbf{a}_z \cdot A_\phi \mathbf{a}_\phi
\end{align*}
\] (1.70)

Equation (1.70) can be written in a matrix form as
We can find these dot products between unit vectors in (1.71). For an example, from Figure 1–15 and Table 1–2, we can express \( \mathbf{a}_r \) in terms of \( \mathbf{a}_\rho \) and \( \mathbf{a}_z \) as
\[
\mathbf{a}_r = (\mathbf{a}_r \cdot \mathbf{a}_\rho) \mathbf{a}_\rho + (\mathbf{a}_r \cdot \mathbf{a}_z) \mathbf{a}_z = \sin \theta \mathbf{a}_\rho + \cos \theta \mathbf{a}_z \tag{1.72}
\]
and then from Table 1–1
\[
\mathbf{a}_z \cdot \mathbf{a}_r = \sin \theta (\mathbf{a}_z \cdot \mathbf{a}_\rho) + \cos \theta (\mathbf{a}_z \cdot \mathbf{a}_z) = \sin \theta \cos \phi \tag{1.73}
\]
Likewise, we can work out other dot products in (1.71). The results are given in Table 1–3.

### Table 1–3 Dot products of unit vectors in Cartesian and spherical coordinate systems

<table>
<thead>
<tr>
<th></th>
<th>( \mathbf{a}_r )</th>
<th>( \mathbf{a}_\theta )</th>
<th>( \mathbf{a}_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{a}_r \cdot )</td>
<td>( \sin \theta \cos \phi )</td>
<td>( \cos \theta \cos \phi )</td>
<td>( -\sin \phi )</td>
</tr>
<tr>
<td>( \mathbf{a}_\theta \cdot )</td>
<td>( \sin \theta \sin \phi )</td>
<td>( \cos \theta \sin \phi )</td>
<td>( \cos \phi )</td>
</tr>
<tr>
<td>( \mathbf{a}_\phi \cdot )</td>
<td>( \cos \theta )</td>
<td>( -\sin \theta )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

Therefore (1.71) becomes
\[
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix} =
\begin{bmatrix}
\sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
\cos \theta & -\sin \theta & 0
\end{bmatrix}
\begin{bmatrix}
A_r \\
A_\theta \\
A_\phi
\end{bmatrix}
\tag{1.74}
\]

or
\[
\begin{align*}
A_x &= A_r \sin \theta \cos \phi + A_\theta \cos \theta \cos \phi - A_\phi \sin \phi \\
A_y &= A_r \sin \theta \sin \phi + A_\theta \cos \theta \sin \phi + A_\phi \cos \phi \\
A_z &= A_r \cos \theta - A_\theta \sin \theta 
\end{align*}
\tag{1.75}
\]

where \( \sin \theta \) and \( \cos \theta \) can be expressed in Cartesian coordinates from (1.43) and (1.64) as
\[
\begin{align*}
\sin \theta &= \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \\
\cos \theta &= \frac{z}{\sqrt{x^2 + y^2 + z^2}}
\end{align*}
\tag{1.76}
\]

Note (1.74) or (1.75) can also be derived from (1.47) and (1.62) by multiplying two transformation matrices in these two equations together. The expressions of \( \sin \phi \) and \( \cos \phi \) in Cartesian coordinates have already been given in (1.48).

Likewise, we can also obtain vector field components in spherical coordinates from Cartesian coordinates as
\[
\begin{bmatrix}
A_x \\
A_y \\
A_z \\
\end{bmatrix}
= 
\begin{bmatrix}
a_x \cdot a_x & a_y \cdot a_x & a_z \cdot a_x \\
a_y \cdot a_y & a_y \cdot a_y & a_z \cdot a_y \\
a_z \cdot a_z & a_y \cdot a_z & a_z \cdot a_z \\
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z \\
\end{bmatrix}
= 
\begin{bmatrix}
sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z \\
\end{bmatrix}
\] (1.77)

or

\[
\begin{align*}
A_x &= A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta \\
A_y &= A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta \\
A_z &= -A_x \sin \phi + A_y \cos \phi
\end{align*}
\] (1.78)

A summary of the unit vectors, the differential lengths, the differential surfaces, and the differential volumes for the three coordinate systems is given in the summary of this chapter (Table 1–4). A summary of the transformations of the variables between coordinate systems is also given in the summary of the chapter (see Table 1–5). Appendix A provides a summary of the vector operations in these three coordinate systems.

**EXAMPLE 1.5**

The Cartesian coordinates for a point P is (1,2,3), find the cylindrical coordinates.

**Solution:**

To find the cylindrical coordinates of P, we can use (1.43)

\[
\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.23
\]

\[
\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) \approx 1.1071 \text{ radian} = 63.43^\circ
\]

\[
z = 3
\]

**MATLAB Solution:**

Although MATLAB has 4 functions cart2pol, cart2sph, pol2cart and sph2cart to convert coordinate variables, the input sequence for cylindrical coordinate is \([\phi, \rho, z]\) rather than \([\rho, \phi, z]\) which follows the right-hand rule and MATLAB uses angle \(\pi / 2 - \theta\) rather than \(\theta\) in spherical coordinates. Therefore we do not recommend to use them. For coordinate transformation, we can easily use formulas in Table to convert from one coordinate system to another. The code for this example can be easily written as follows:

\[
x = 1;
y = 2;
z = 3;
rho = \text{sqrt}(x^2 + y^2)
\]

\[
\phi = \text{atan2}(y, x) \quad \% \text{Note: we recommend to use atan2 rather than atan}
\%
\text{since atan2 can provide the correct quadrant of the angle}
\]

\[
z = z
\]
However, we recommend students to create your own functions to convert between different coordinates. This example can also be done by a self defined function to convert from Cartesian to cylindrical coordinate. The code is as follows:

```matlab
>> Pcar = [1, 2, 3];
>> Pcyl = car2cyl(Pcar)
Pcyl =
    2.2361    1.1071    3.0000
```

where the self-defined function `car2cyl` is written as

```matlab
function Pcyl = car2cyl(Pcar)
x = Pcar(1);
y = Pcar(2);
z = Pcar(3);
rho  =  sqrt(x^2+y^2);
phi  =  atan2(y, x) ;
Pcyl(1) = rho;
Pcyl(2) = phi;
Pcyl(3) = z;
end
```

The source code is listed in `ex105.m`.

**EXAMPLE 1.6**

Express the vector field \( \mathbf{A} = 5\sin \phi \mathbf{a}_\rho + 7\rho \mathbf{a}_\phi - 3\cos \phi \mathbf{a}_z \) in Cartesian coordinates.

**Solution:**

From equation (1.65), we have

\[
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
5\sin \phi \\
7\rho \\
-3\cos \phi
\end{bmatrix} =
\begin{bmatrix}
5\sin \phi \cos \phi - 7\rho \sin \phi \\
5\sin^2 \phi + 7\rho \cos \phi \\
-3\cos \phi
\end{bmatrix}
\]

We still need to convert variables to \( x, y, z \). This can be done using (1.43) and (1.49):

\[
\rho = \sqrt{x^2 + y^2}, \sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}
\]

Therefore,

\[
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix} =
\begin{bmatrix}
5\frac{xy}{x^2 + y^2} - 7y \\
5\frac{y^2}{x^2 + y^2} + 7x \\
-3\frac{x}{\sqrt{x^2 + y^2}}
\end{bmatrix}
\]
MATLAB Solution:

We can use MATLAB symbolic math to work out this example. The source code ex106.m is listed as follows:

```matlab
syms rho theta phi x y z
T = [cos(phi) -sin(phi) 0; sin(phi) cos(phi) 0; 0 0 1];
Acyl = [5*sin(phi); 7*rho; -3*cos(phi)];
Acar = T*Acyl;
Acar = subs(Acar, {rho, phi}, {sqrt(x^2+y^2), atan(y,x)});
Acar = simplify(Acar)
```

1.6 Integral Relations for Vectors

We will find that certain integrals involving vector quantities are important when describing the material presented later in this text. These integrals will be useful initially in deriving vector operations and later in gaining an understanding of electromagnetic fields. The fact that a field’s behavior could depend on its local position should not be too surprising. Recall the effects of a change in the gravitational field while watching the astronauts walking into a ground-based spacecraft, then floating within the spacecraft in atmosphere. The integrals we will focus on are listed in Table 1–4.

<table>
<thead>
<tr>
<th>TABLE 1–4 Integrals of Vector Fields and Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line integral of a vector field ( \mathbf{F} ) along a prescribed path from the location ( a ) to the location ( b ).</td>
</tr>
<tr>
<td>Surface integral of a vector field ( \mathbf{A} ) through a surface ( S ).</td>
</tr>
<tr>
<td>Volume integral of a density ( \rho ), over the volume ( V ).</td>
</tr>
</tbody>
</table>

1.6.1 Line Integral

Let us first examine a line integral. One possible application of this integral would be to compute the work \( W \) that would be required to push the cart with a force \( \mathbf{F} \) from point \( a \) to point \( b \) along a prescribed path as shown in Figure 1–16. This path could be dictated by metallic rails underneath the cart. The line integral is written as

\[
\int_a^b \mathbf{F} \cdot d\mathbf{l} \tag{1.74}
\]
The cart is constrained to move along the prescribed path from points $a$ to $b$.

The differential length element $dl$ can be written in the three orthogonal coordinate systems, and these were included in (1.34), (1.39) and (1.54). The limits $a$ and $b$ determine the sign of the integration, i.e., $+$ or $-$. This integral states that no work is expended in moving the cart if the direction of the force applied to the object is perpendicular to the path of the motion. If we were to push the cart completely around the path so it returned to the original point, we would call this a *closed line integral* and indicate it with a circle at the center of the integral sign as in (1.75) below:

$$\oint_C \mathbf{F} \cdot d\mathbf{l}$$  \hspace{1cm} (1.75)

To illustrate this point, let us calculate the work required to move the cart along path 1 as indicated in Figure 1–17 against a force field $\mathbf{F}$ where

$$\mathbf{F} = 3xy \mathbf{a}_x + 4xy \mathbf{a}_y$$  \hspace{1cm} (1.76)
In this example, we are able to specify a numerical value for one of the variables along each segment of the total path, since each path is chosen to be parallel to an axis of the Cartesian coordinate system. This is not always possible, and one of the variables may have to be specified in terms of the other variable, or these dependent variables may be a function of another independent parameter, for example, time. In this example, the work is found using the line integral. This integral will consist of two terms, since the path of integration is initially parallel to the $x$ axis and then parallel to the $y$ axis. In the first term, the incremental change in $y$ is zero, hence $dy = 0$ and the differential length becomes $dl = dx a_x$. Similarly, $dl = dy a_y$ in the second integration since $dx = 0$. Therefore, we write

$$\Delta W = \int_{(1,1)}^{(4,2)} F \cdot dl = 3 \frac{x^2}{2} + 16 \frac{y^2}{2} = \frac{93}{2} \quad (1.77)$$

We could return from point $b$ back to point $a$ along the same path that we followed earlier or along a different path—say, path 2 in Figure 1–17. We calculate the work along this new path. The differential length $dl$ remains the same even though there is a change of direction in the integration. The limits of the integration will specify the final sign to be encountered from the integration.
The total work required to move the cart completely around this closed path is not equal to zero! A closed path is defined as any path that returns us to the original point. In Figure 1–17, the cart could have been pushed completely around the loop. There may or may not be something enclosed within the closed path. In order to emphasize this point, think of walking completely around the perimeter of a green on a golf course. This would be an example of a closed path. The entity that would be enclosed within this path and rising above the ground would be the flag. If the closed line integral over all possible paths were equal to zero, then the vector \( \mathbf{F} \) would belong to a class of fields that are called **conservative fields**. The example that we have just encountered would correspond to the class of **nonconservative fields**. Both conservative and nonconservative fields will be encountered in electromagnetics.

As electrical and computer engineers, you have already encountered this integral in the first course in electrical circuits without even knowing it. If we sum up the voltage drops around a closed loop, we find that they are equal to zero. This is, of course, just one of Kirchhoff’s laws. For the cases that we have encountered in that early circuit’s course, this would be an example of a conservative field.

**EXAMPLE 1.7**

Calculate the work \( \Delta W \) required to move the cart along the closed path in Figure 1–17 if the force field is \( \mathbf{F} = 3x\mathbf{a}_x + 4y\mathbf{a}_y \).

**Solution:**

The closed line integral is given by the sum of four integrals.

\[
\Delta W = \oint \mathbf{F} \cdot d\mathbf{l} = \int_{(1, 1)}^{(4, 1)} (3x\mathbf{a}_x + 4y\mathbf{a}_y) \cdot dx \mathbf{a}_x + \int_{(4, 2)}^{(1, 1)} (3x\mathbf{a}_x + 4y\mathbf{a}_y) \cdot dy \mathbf{a}_y =
\]

\[
= \int_{1}^{4} 3x dx + \int_{2}^{1} 4y dy + \int_{1}^{4} 3x dx + \int_{2}^{1} 4y dy
\]

\[
= \frac{3x^2}{2} \bigg|_{1}^{4} + \frac{4y^2}{2} \bigg|_{2}^{1} + \frac{3x^2}{2} \bigg|_{1}^{4} + \frac{4y^2}{2} \bigg|_{2}^{1} = 0
\]

In this case, the force field \( \mathbf{F} \) is a conservative field.

**MATLAB Solution:**

In MATLAB, 1D integration can be symbolically carried out using MATLAB function \texttt{int}.

The solution can be easily written as:

\[
syms x y
Fx = 3*x;
Fy = 4*y;
dW = int(Fx, x, 1, 4) + int(Fy, y, 1, 2) + int(Fx, x, 4, 1) + int(Fy, y, 2, 1)
\]

The code is listed in EX107.m.
EXAMPLE 1.8
Calculate the work $\Delta W$ required to move the cart along the circular path from point A to point B if the force field is $\mathbf{F} = 3xy\mathbf{a}_x + 4x\mathbf{a}_y$.

![Figure for Example 1.8](image)

Solution:
The integral can be performed in Cartesian coordinates or in cylindrical coordinates. In Cartesian coordinates, we write

$$\mathbf{F} \cdot d\mathbf{l} = (3xy\mathbf{a}_x + 4x\mathbf{a}_y) \cdot (dx\mathbf{a}_x + dy\mathbf{a}_y) = 3xy\, dx + 4x\, dy$$

The equation of the circle is $x^2 + y^2 = 4^2$. Hence

$$\int_{\mathcal{A}}^{\mathcal{B}} \mathbf{F} \cdot d\mathbf{l} = \int_{0}^{4\pi} 3x\sqrt{16 - x^2} \, dx + \int_{0}^{4\pi} 4\sqrt{16 - y^2} \, dy$$

$$= -\left(16 - x^2\right)^{3/2} \bigg|_{0}^{4\pi} + 4\left(\frac{y^2}{2} - \sqrt{16 - y^2} + 8\sin^{-1}\left(\frac{y}{4}\right)\right) \bigg|_{0}^{4\pi} = -64 + 16\pi$$

In cylindrical coordinates, we convert $\mathbf{F}$ in cylindrical coordinates at first. From (1.50) and (1.42), we have

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3xy \\ 4x \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\rho^2 \cos \phi \sin \phi \\ 4\rho \cos \phi \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\rho^2 \cos^2 \phi \sin \phi + 4\rho \cos \phi \sin \phi \\ -3\rho^2 \cos \phi \sin^2 \phi + 4\rho \cos^2 \phi \\ 0 \end{bmatrix}$$

Since the integral is to be performed along the indicated path where only the angle $\phi$ is changing, we have $d\rho = 0$ and $dz = 0$. Also $\rho = 4$. Therefore

$$\mathbf{F} \cdot d\mathbf{l} = (F_\rho \mathbf{a}_\rho + F_\phi \mathbf{a}_\phi) \cdot (d\rho \mathbf{a}_\rho + \rho d\phi \mathbf{a}_\phi) = (F_\phi \rho) \bigg|_{\rho = 4} \, d\phi = -64(3 \sin^2 \phi \cos \phi - \cos^2 \phi)$$

The integral becomes
\[
\int_S \mathbf{F} \cdot d\mathbf{a} = -64 \int_0^{\pi/2} (3 \sin^2 \phi \cos \phi - \cos^2 \phi) d\phi \\
= -64 \left( \frac{3}{3} \sin \phi - \frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) \bigg|_0^{\pi/2} = -64 + 16\pi
\]

The results of the two calculations are identical as should be expected.

**MATLAB Solution:**

In Cartesian coordinates, the MATLAB code is listed as follows:

```matlab
syms x y
Fx = 3*x*y;
Fy = 4*x;
Fx = subs(Fx,y,sqrt(16-x^2));
Fy = subs(Fy,x,sqrt(16-y^2));
dW = int(Fx,x,4,0) + int(Fy,y,0,4)
```

In Cylindrical coordinates, the MATLAB code is listed as follows:

```matlab
syms x y rho phi
Fcart = [3*x*y; 4*x];
Fcart = subs(Fcart, {x,y},{rho*cos(phi), rho*sin(phi)});
Fcyl = [cos(phi) sin(phi); -sin(phi) cos(phi)]*Fcart;
Fphi = Fcyl(2);
Fphi_rho = subs(Fphi*rho, rho, 4);
dW = int(Fphi_rho, phi, 0, pi/2)
```

The source code is listed in ex108.m

### 1.6.2 Surface Integral

Another integral, that will be encountered in the study of electromagnetic fields is the surface integral, which is written as

\[
\iint_S \mathbf{A} \cdot d\mathbf{s} 
\]

(1.79)

where \( \mathbf{A} \) is the vector field and \( d\mathbf{s} \) is the differential surface area. The differential surface areas for the three coordinate systems are given in (1.35), (1.40) and (1.55). A field flowing through a surface is shown in Figure 1–18 for an arbitrary surface. The vector \( \mathbf{F} \) at this stage could represent a fluid flow. In some sense, the loop monitors the flow of the field.
A surface integral for an arbitrary surface. At the particular location of the loop, the component of \( \mathbf{A} \) that is tangent to the loop does not pass through the loop. The scalar product \( \mathbf{A} \cdot ds \) eliminates this contribution.

The differential surface element is, by definition, a vector since a direction is associated with it. The vector orientation of \( ds \) is in the direction that is normal to the surface outward. For a closed surface, this is the obvious direction. However, for a nonclosed surface such as a plane or our golfing green, the user must specify it in the absence of any obvious outward direction. Using the “right hand rule” convention, we rely on which way the thumb points if the fingers of the right hand follow the perimeter of the surface counterclockwise. A person standing atop the golfing green would observe a different direction than an individual underneath the green.

The surface integral allows us to ascertain the amount of the vector field \( \mathbf{A} \) that is passing through a surface \( \Delta s \), which has a differential surface element \( ds \). This vector field is frequently called a flux. A vector \( \mathbf{F} \) that is directed tangential to the surface will have the scalar product \( \mathbf{A} \cdot ds = 0 \) — i.e., the vector \( \mathbf{F} \) does not pass through the surface.

If we integrated the vector field over an entire closed surface, the notation

\[
\iint A \cdot ds
\]

is employed. As we will see later, this closed surface integral can be either: greater than zero, equal to zero, or less than zero depending on what is contained within the
enclosed volume. For the cubical surface shown in Figure 1–19, there are six vectors $\mathbf{ds}$ associated with the six differential surfaces. The vectors $\mathbf{ds} = \mathbf{dxdy} \mathbf{a}_z$ and $\mathbf{ds} = \mathbf{dxdy} (-\mathbf{a}_z)$ for the two surfaces that are perpendicular to the $z$ axis and are opposite one another also have vectors oriented in opposite directions. The other four differential surfaces are similarly defined.

**FIGURE 1–19**
There are six differential surface vectors associated with a cube. The vectors are directed outwards.

**EXAMPLE 1.9**

Assume that vector field $\mathbf{A} = \mathbf{A}_0/r^2 \mathbf{a}_r$ exists in a region surrounding the origin of a spherical coordinate system. Find the value of the closed-surface integral $\iint \mathbf{A} \cdot \mathbf{ds}$ over the unit sphere.

Figure for Example 1.9
Solution:

The closed-surface integral is given by

\[ \iiint A \cdot ds = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left( \frac{A_r}{r^2} \right) \cdot (r^2 \sin \theta \, d\theta \, d\phi) = 4\pi A_0 \]

In this integral, we have used the differential surface area in spherical coordinates that has a unit vector \( a_r \). If the vector \( A \) had any additional components directed in the \( a_\theta \) or \( a_\phi \) directions, their contribution to this surface integral would be zero, since the scalar product of these terms will be equal to zero.

MATLAB Solution:

The MATLAB code (ex109.m) is listed as follows:

```matlab
syms r theta phi A0
Ar = A0/r^2;
integrand = Ar*r^2*sin(theta);
SurfaceIntegral = int(int(integrand, theta, 0,pi), phi, 0, 2*pi);
```

EXAMPLE 1.10

Assume that the vector field \( A = 5xya_x + xy^2a_y + 4za_z \) is defined in a region \( 1 \leq x \leq 3, -2 \leq y \leq 4 \) and \(-1 \leq z \leq 2\). Find the value of the closed-surface integral \( \iiint A \cdot ds \) over the surface of this region.

Solution:

The region has six surfaces (see Fig. 1-19). On the front surface, \( x = 3 \) and \( A_x \) is the only component perpendicular to that surface. Therefore,

\[ A \cdot ds \bigg|_{\text{front}} = A_x \bigg|_{x=3} \, dy \, dz = 5 \times 3 \, y \, dy \, dz = 15 \, y \, dy \, dz \]

\[ \int \int \int A \cdot ds \bigg|_{\text{front}} = \int_{-1}^{2} \int_{-2}^{4} 15 \, y \, dy \, dz = \int_{-1}^{2} 15 \left( \frac{4^2 - (-2)^2}{2} \right) \, dz = 270 \]

On the back surface, \( x = 1 \) and \( A_x \) is the only component perpendicular to that surface. Therefore,

\[ A \cdot ds \bigg|_{\text{back}} = A_x \bigg|_{x=1} \, (-dy \, dz) = -5 \times 1 \, y \, dy \, dz = -5 \, y \, dy \, dz \]

\[ \int \int \int A \cdot ds \bigg|_{\text{back}} = \int_{-1}^{2} \int_{-2}^{4} -5 \, y \, dy \, dz = -\int_{-1}^{2} 5 \left( \frac{4^2 - (-2)^2}{2} \right) \, dz = -90 \]

On the right surface, \( y = 4 \) and \( A_y \) is the only component perpendicular to that surface. Therefore,
\[ \mathbf{A} \cdot ds_{\text{right}} = A_y \mid_{y=4} \, dxdz \\
= x \times 4^2 \, dxdz = 16 \, dxdz \]

\[ \iint \mathbf{A} \cdot ds_{\text{right}} = \int_{-1}^{1} \int_{-1}^{3} 16 \, dxdz = \int_{-1}^{3} \frac{3^2 - 1^2}{2} \, dz = 192 \]

On the left surface, \( y = -2 \) and \( A_y \) is the only component perpendicular to that surface. Therefore,

\[ \mathbf{A} \cdot ds_{\text{left}} = A_y \mid_{y=-2} \, (-dxdz) \\
= x \times (-2)^2 \, (-dxdz) = -4 \, dxdz \]

\[ \iint \mathbf{A} \cdot ds_{\text{left}} = \int_{-1}^{3} \int_{-1}^{1} -4 \, dxdz = \int_{-1}^{1} -\frac{3^2 - 1^2}{2} \, dz = -48 \]

On the top surface, \( z = 2 \) and \( A_z \) is the only component perpendicular to that surface. Therefore,

\[ \mathbf{A} \cdot ds_{\text{top}} = A_z \mid_{z=2} \, dxdy \\
= 4 \times 2 \, dxdy = 8 \, dxdy \]

\[ \iint \mathbf{A} \cdot ds_{\text{top}} = \int_{1}^{3} \int_{-1}^{4} 8 \, dxdy = 96 \]

On the bottom surface, \( z = -1 \) and \( A_z \) is the only component perpendicular to that surface. Therefore,

\[ \mathbf{A} \cdot ds_{\text{bot}} = A_z \mid_{z=-1} \, (-dxdy) \\
= 4 \times (-1) \, (-dxdy) = 4 \, dxdy \]

\[ \iint \mathbf{A} \cdot ds_{\text{bot}} = \int_{-2}^{3} \int_{-1}^{4} 4 \, dxdy = 48 \]

The closed surface integral can be obtained by the summation of the above six surface integrals as:

\[ \iiint A \cdot ds = 270 - 90 + 192 - 48 + 120 + 60 = 468 \]

**MATLAB Solution:**

The MATLAB code (ex110.m) is listed as follows:

```matlab
syms x y z
A = [5*x*y, x*y^2, 4*z];
xmin = 1;
xmax = 3;
ymin = -2;
ymax = 4;
zmin = -1;
zmax = 2;
front = subs(A(1),x,xmax);
intFront = int(int(front,y,ymin,ymax),z,zmin,zmax);
back = - subs(A(1),x,xmin);
intBack = int(int(back,y,ymin,ymax),z,zmin,zmax);
right = subs(A(2),y,ymax);
```
\[ \text{intRight} = \text{int(int(right,x,xmin,xmax),z,zmin,zmax);} \]
\[ \text{left} = \text{- subs(A(2),y,ymin);} \]
\[ \text{intLeft} = \text{int(int(left,x,xmin,xmax),z,zmin,zmax);} \]
\[ \text{top} = \text{subs(A(3),z,zmax);} \]
\[ \text{intTop} = \text{int(int(top,x,xmin,xmax),y,ymin,ymax);} \]
\[ \text{bottom} = \text{- subs(A(3),z,zmin);} \]
\[ \text{intBottom} = \text{int(int(bottom,x,xmin,xmax),y,ymin,ymax);} \]
\[ \text{SurfaceIntegral} = \text{intFront + intBack + intRight + intLeft + intTop + intBottom} \]

### 1.6.3 Volume Integral

Finally, we will encounter various volume integrals of scalar quantities, such as a volume charge density \( \rho_v \). A typical integration would involve the computation of the total charge or mass in a volume if the volume charge density or mass density were known. It is written as

\[ Q = \iiint\rho_v \, dv \quad (1.81) \]

The differential volumes for the three coordinate systems are given in Table 1–1. This will be demonstrated with an example.

**EXAMPLE 1.11**
Find the volume of a cylinder that has a radius \( a \) and a length \( l \).

![Figure for Example 1.11](image)

**Solution:**
The volume of a cylinder is calculated to be

\[ V = \iiint r \, dv = \int_0^l \int_0^{2\pi} \int_0^a \rho \, dr \, d\phi \, dz = \pi a^2 l \]

**MATLAB Solution:**
The MATLAB code (ex111.m) is listed as follows:

```matlab
syms rho phi z a l
func = rho;
V = int(int(int(func, rho, 0, a), phi, 0, 2*pi), z, 0, l)
```

**EXAMPLE 1.12**
Find the total charge within a volume defined by

\[ 1 \leq r \leq 5, \ 0 \leq \theta \leq \pi / 2, \ \pi / 4 \leq \phi \leq 3\pi / 4 \]
if the charge density is given as: \( \rho_v = r \cos \phi \)

**Solution:**

\[
Q = \iiint \rho_v dv = \int_0^\pi \int_{\pi/6}^{\pi/2} \int_1^5 r \cos \phi \cdot r^2 \sin \theta \, dr \, d\theta \, d\phi = -39\sqrt{6}
\]

**MATLAB Solution:**

The MATLAB code (ex112.m) is listed as follows:

```matlab
syms r theta phi
rho_v = r*cos(phi);
func = rho_v*r^2*sin(theta);
Q = int(int(int(func, r, 1, 5), theta, pi/6, pi/2), phi, pi/4, pi)
```

1.7 Differential Relations for Vectors

In addition to the integral relations for vectors, there are also differential operations that will be encountered frequently in our journey through electromagnetic theory. Each of these differential operators can be interpreted in terms of physical phenomena. We will concentrate on vector operations in Cartesian coordinates. In addition, the operations in cylindrical and spherical coordinates are included in this discussion. The three vector operations are given in Table 1–5. A summary of these vector operations in the three coordinate systems is found in Appendix A.

1.7.1 Gradient

It is possible to methodically measure scalar quantities (such as a temperature) at various locations in space. From these data, it is possible to connect the locations where the temperatures are the same. When placed on a graph in a two dimensional plot, these equitemperature contours are useful in interpreting various effects. This could include determining the magnitude and direction where the most rapid changes occur or ascertaining the amount of heat that will flow in a particular direction. This could be useful in planning a ski trip or beach vacation. The gradient of the scalar quantity (which in this case is the temperature) allows us to compute the magnitude and the direction we should follow to find the maximum spatial rate of change in the scalar quantity to attain the desired conditions.

**TABLE 1–5** Three important differential operations expressed with the Del operator

<table>
<thead>
<tr>
<th><strong>Gradient of a scalar function</strong></th>
<th>( \nabla V )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Divergence of a vector field</strong></td>
<td>( \nabla \cdot \mathbf{A} )</td>
</tr>
<tr>
<td><strong>Curl of a vector field</strong></td>
<td>( \nabla \times \mathbf{A} )</td>
</tr>
</tbody>
</table>

In Figure 1–20, we sketch two equipotential surfaces in space; the potential of one surface is arbitrarily chosen to have the value \( V \), and the potential of the other surface is \( V + \Delta V \). Point 1 is located on the first surface. The unit vector \( \mathbf{a}_n \) that is normal to this surface at \( P_1 \) intersects the second surface at point \( P_2 \). The magnitude of the distance between these two points is \( n \). Point \( P_3 \) is another point on the second surface, and the vector distance between \( P_1 \) and \( P_3 \) is \( \Delta l \). The unit vector from \( P_1 \) to \( P_3 \)
is \( \mathbf{a}_l \). The angle between the two vectors is \( \zeta \). The distance \( \Delta l \) is greater than \( \Delta n \). Therefore,

\[
\frac{\Delta V}{\Delta n} \geq \frac{\Delta V}{\Delta l} \tag{1.82}
\]

This allows us to define the first differential operation.

**FIGURE 1–20**

Equipotential surfaces in space.

The gradient is defined as the vector that represents both the magnitude and the direction of the maximum spatial rate of increase of a scalar function. It depends on the position where the gradient is to be evaluated, and it may have different magnitudes and directions at different locations in space. Referring to Figure 1–20, we write the gradient as

\[
\nabla \equiv \lim_{\Delta n \to 0} \nabla \frac{\Delta V}{\Delta n} = \mathbf{a}_n
\tag{1.83}
\]

In writing (1.83), we have used the common notation of replacing grad with the symbol \( \nabla \) (“del”). In addition, we have assumed that the separation distance between the two surfaces is small, and let \( \Delta n \to dn \), which is indicative of a derivative.

The definition of a directional derivative is self-explanatory. In Figure 1–20 we want it in the \( \mathbf{a}_l \) direction, and we write

\[
\frac{\Delta V}{\Delta l} \mathbf{a}_l \approx \frac{dV}{dl} \mathbf{a}_l \tag{1.84}
\]

where we have again let \( \Delta l \to dl \). Using the chain rule, we find that

\[
\frac{dV}{dl} = \frac{dV}{dn} \frac{dn}{dl} = \frac{dV}{dn} \cos \zeta = \frac{dV}{dn} \mathbf{a}_n \cdot \mathbf{a}_l = \nabla \cdot \mathbf{a}_l \tag{1.85}
\]
We realize that the directional derivative in the \( \mathbf{a}_i \) direction is the projection of the gradient in that particular direction. Equation (1.85) can be written as
\[
d V = \nabla \mathbf{V} \cdot d\mathbf{l} = \nabla \mathbf{V} \cdot d\mathbf{l} \tag{1.86}
\]

The gradient of the scalar function \( V(x,y,z) \) is given in Cartesian coordinates as
\[
\nabla V = \frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z \tag{1.87}
\]

This equation prompts us to define the “del operator” in Cartesian coordinates (only) as
\[
\nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \tag{1.88}
\]

The gradient of the scalar function \( V(\rho,\phi,z) \) in cylindrical coordinates is
\[
\nabla V = \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z \tag{1.89}
\]

The gradient of the scalar function \( V(r,\theta,\phi) \) in spherical coordinates is
\[
\nabla V = \frac{\partial V}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi \tag{1.90}
\]

MATLAB also provides the capability of performing the gradient operation. In order to use this command, we must first calculate the contours that connect the points that have the same value.

**EXAMPLE 1.13**

Assume that there exists a surface that can be modeled with the equation \( z = e^{-(x^2+y^2)} \).

Calculate \( \nabla z \) at the point \((x = 0, y = 0)\). In addition, use MATLAB to illustrate the profile and to calculate and plot this field.

**Solution:**
\[
\nabla z = -2xe^{-(x^2+y^2)}\mathbf{a}_x - 2ye^{-(x^2+y^2)}\mathbf{a}_y . \text{ At the point } (x = 0, y = 0), \nabla z = 0.
\]

**MATLAB Solution:**

We can symbolically work out this example. The code is written as follows:

```matlab
syms x y
z = exp(-(x^2+y^2))
Del_z = [diff(z,x) diff(z,y)]
subs(Del_z, {x,y},{0,0})
```

Besides, MATLAB function gradient can be used to numerically find the gradient of a function. Using MATLAB, the function is illustrated in Figure (a). The contours with the same value are connected together, and the resulting field is indicated in (b). The length of the vectors and their orientation clearly indicate the distribution of the field in space. The commands `gradient`, `contour`, and `quiver` have been employed to create the figure. The source code is provided in ex113.m.
1.7.2 Divergence

The second vector derivative that should be reviewed is the divergence operation. The divergence operator is useful in determining if there is a source or a sink at locations in space where a vector field exists. For electromagnetic fields, these sources and sinks will turn out to be positive and negative electrical charges. This region could also be situated in a river where water would be flowing, as shown in Figure 1–21. This could be a very porous box that contained either a drain or faucet that was connected with an invisible hose to the shore, where the fluid could either be absorbed or from which it could be extracted.
The divergence of a vector that applies at a point is defined by the expression

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_{\partial \Delta v} \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

(1.91)

The symbol $\oint_{\partial \Delta v} \mathbf{A} \cdot d\mathbf{s}$ indicates an integral over the entire closed surface that encloses the volume $\Delta v$. The point where the divergence is evaluated is within the volume $\Delta v$, and the surface for the closed-surface integral is the surface that surrounds this volume. As we will see, the application of the “$\nabla \cdot$” notation (where $\nabla$ is the del operator) will help us in remembering the terms that appear in the operation.

We evaluate the surface integral over the pair of constant $x$ surfaces only; the remaining surface integrals are similar and need not be repeated. The only component of the vector $\mathbf{A}$ involved is the component $A_x$, due to the dot products with $d\mathbf{s}$. We use a Taylor series expansion of $A_x$ about the midpoint of the volume. Recall that an arbitrary function $f(x)$ has the Taylor series

$$f(x) = f(x_0) + \frac{\partial f(x)}{\partial x} \bigg|_{x_0} (x - x_0) + \frac{\partial^2 f(x)}{\partial x^2} \bigg|_{x_0} (x - x_0)^2 + \cdots$$

(1.92)
We will retain only the first two terms of the series, since in our application \( x \) will approach \( x_0 \) in the limit, and hence higher powers in \((x - x_0)\) will be very small. Furthermore, we will assume that the functions to be expanded are sufficiently smooth that indicated derivatives pose no problems.

**Figure 1–22** depicts the two surfaces over which we integrate. Since the volume is very small, the quantity \( A_x \) may be assumed constant over the surfaces. The surface integral in this case yields \( A_x \) multiplied by the surface area \( \Delta y \Delta z \). Note that the normal directions for the two surfaces differ in sign, as shown in the figure. Thus, the surface integral over the two surfaces is given by

\[
\int (A_x a_x) \, ds \approx (A_x |_{x+\Delta x}) \cdot (\Delta y \Delta z a_x) + (A_x |_{x}) \cdot (-\Delta y \Delta z a_x)
\]

\[
(\Delta x \Delta y \Delta z)
\]

Note that if \( A_x \) doesn’t change with \( x \), then the integral is zero. This corresponds to the situation where all fluid flowing in through one surface exits the other; there is no source nor sink in the intermediate region.

Using the result of equation (1.93) in the definition (1.91) of the divergence gives the result

\[
\nabla \cdot (A_x a_x) = \lim_{\Delta x \to 0} \frac{(\partial A_x)}{\partial x} \frac{(\Delta x \Delta y \Delta z)}{(\Delta x \Delta y \Delta z)} = \frac{\partial A_x}{\partial x}
\]
where the volume in question is \( \Delta v = \Delta x \Delta y \Delta z \). We may repeat this analysis for the remaining pairs of faces of the cube and sum the results to obtain the well-known result

\[
\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}
\]

(1.95)

Once again, in Cartesian coordinates this suggests the definition of the del operator, this time in a scalar product with a vector. However, be cautious: the del operator assumes this simple form only in Cartesian coordinates, and it is an operator, not a vector. In other words, \( \nabla \cdot \mathbf{A} \neq \mathbf{A} \cdot \nabla \)!

**EXAMPLE 1.14**

Find the divergence of the vector \( \mathbf{A} = 3x \mathbf{a}_x + xy^2 \mathbf{a}_y - 2xy e^{-z} \mathbf{a}_z \) at the point \( (1, -1, 2) \).

**Solution:**

Using equation (1.95), we find

\[
\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} (3x) + \frac{\partial}{\partial y} (xy^2) - \frac{\partial}{\partial z} (2xy e^{-z})
\]

\[
= 3 + 2xy + 2xy e^{-z}
\]

At the point \( (1, -1, 2) \) this gives \( \nabla \cdot \mathbf{A} = 3 - 2 - 2e^{-2} = 0.7293 \).

**MATLAB Solution:**

The code (ex114.m) is listed as follows:

```matlab
syms x y z
A = [3*x*x y^2 -2*x*y*exp(-z)];
divA = diff(A(1),x) + diff(A(2),y) + diff(A(3),z)
subs(divA, {x,y,z}, {1,-1,2})
```

We have found the divergence of a vector and we can suggest a physical interpretation of it. If the divergence of a vector \( \mathbf{A} \) is equal to zero, then there are neither sources to create the vector \( \mathbf{A} \) nor sinks to absorb it at that location. In this case, everything that enters the volume will leave it unscathed. If the divergence of a vector is greater than or less than zero, then there is either a source or a sink for the vector at that location; there is a net divergence or convergence of flux. In the context of electromagnetic theory, the integral of the divergence of a vector field determines whether positive or negative electric charges exist within a volume (in the case of electric fields), or whether magnetic charges or “magnetic monopoles” exist in the region (for the case of magnetic fields, no such charges have been detected in nature).

We derived the divergence in Cartesian coordinates. The extension to cylindrical and spherical coordinates is similar. In cylindrical coordinates, we find (Appendix A)

\[
\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}
\]

(1.96)

In spherical coordinates, we have

\[
\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}
\]

(1.97)
EXAMPLE 1.15
Find the divergence of the vector field \( \mathbf{A} = \rho e^{-\rho/\alpha} \mathbf{a}_\rho \), where \( \alpha \) is a constant.

**Solution:**
From equation (1.96), we have
\[
\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho^2 e^{-\rho/\alpha} \right) = \frac{1}{\rho} \left[ 2 \rho e^{-\rho/\alpha} - 2 \rho^2 \left( \frac{\rho}{\alpha} \right) e^{-\rho/\alpha} \left( \frac{1}{\alpha} \right) \right] 
\]
\[
= 2 e^{-\rho/\alpha} \left[ 1 - \left( \frac{\rho}{\alpha} \right)^2 \right]
\]

**MATLAB Solution:**
We can symbolically work out this example. The code is listed as follows:

```matlab
syms rho phi z alpha
A = [rho*exp(-rho/alpha)^2 0 0];
divA = (1/rho)*diff(rho*A(1), rho) + (1/rho)*diff(A(2), phi) + diff(A(3),z);
divA = simplify(divA)
```

By application of the MATLAB `divergence` function, we can also numerically find the divergence of \( \mathbf{A} \). The plot of the vector field \( \mathbf{A} \) using the `quiver` function is shown in Figure (a), and the contours of the divergence, \( \nabla \cdot \mathbf{A} \), are presented in Figure (b). The source code is listed in ex115.m.

(a)

(b)

MATLAB Figure for Example 1.15
From the definition of the divergence (1.91), we can also find a useful relation between a volume integral of the vector’s divergence and the integral of the vector field over the closed surface enclosing the volume \( \Delta v \). This can be obtained from the following “hand-waving” argument. From (1.91), we write

\[
\iiint_D \mathbf{A} \cdot d\mathbf{s} \approx (\nabla \cdot \mathbf{A}) \Delta v \approx \iiint_{\Delta v} (\nabla \cdot \mathbf{A}) d\mathbf{v}
\]  

(1.98)

In passing from the second term that appears in the definition of the divergence to the integral in the third term, we have let the volume \( \Delta v \) be so small that the volume integral of the vector’s divergence is approximately equal to the product of the volume and the divergence.

Equating the two terms involving the integrals and replacing the approximately equal symbol with the equal sign, we obtain the **Divergence Theorem**.

\[
\iiint_D \mathbf{A} \cdot d\mathbf{s} = \iiint_{\Delta v} (\nabla \cdot \mathbf{A}) d\mathbf{v}
\]  

(1.99)

The integral on the right-hand side is over the surface enclosing the volume. This theorem will prove very useful in later developments regarding electromagnetic fields, since it allows us to move easily between a volume integral and a closed-surface integral. It is also known as **Gauss’s Theorem**.

**EXAMPLE 1.16**

Evaluate both sides of the divergence theorem for the vector field

\[
\mathbf{A} = 5xy \mathbf{a}_x + xy^2 \mathbf{a}_y + 4z \mathbf{a}_z
\]

defined in the region \( 1 \leq x \leq 3, -2 \leq y \leq 4 \) and \( -1 \leq z \leq 2 \).

**Solution:**

Using equation (1.95), we find

\[
\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} (5xy) + \frac{\partial}{\partial y} (xy^2) + \frac{\partial}{\partial z} (4z)
\]

\[
= 5y + 2xy + 4
\]

The volume integral is given by

\[
\iiint_V \nabla \cdot \mathbf{A} \, dv = \int_{-1}^{3} \int_{-2}^{4} \int_{-1}^{2} (5y + 2xy + 4) \, dx \, dy \, dz = 468
\]

The closed-surface integral has already been calculated in Example 1.10. We have

\[
\iiint_S \mathbf{A} \cdot d\mathbf{s} = 468
\]

As we expected, the two answers are the same.

**MATLAB Solution:**

The volume integral part is given as follows:

```matlab
syms x y z
A = [5*x*y x*y^2 4*z];
divA = diff(A(1),x) + diff(A(2),y) + diff(A(3),z);
volume_integral = int(int(int(divA, x, 1, 3), y, -2, 4), z, -1, 2)
```

The closed surface integral part is the same as Example 1.10.
1.7.3 Curl

The curl is a vector operation that can be used to determine whether there is a rotation associated with a vector field. This is visualized most easily by considering the experiment of inserting a small paddle wheel in a flowing river as shown in Figure 1–23. If the paddle wheel is inserted in the center of the river, it will not rotate since the velocity of the water a small distance on either side of the center will be the same. However, if the paddle wheel was situated near the edge of the river, it would rotate since the velocity near the edge will be less than in a region further from the edge. Note that the rotation will be in the opposite directions at the two edges of the river.

The curl operation determines both the sense and the magnitude of the rotation.

**FIGURE 1–23**
The paddle wheels inserted in a river will rotate if they are near the edges, since the river velocity just at the edge is zero. The wheel at the center of the river will not rotate.

The curl of a vector $A$ gives a vector result. The $a_n$ component of the curl is defined by

$$\lim_{\Delta s \to 0} \frac{\oint A \cdot dl}{\Delta s}$$

Here, the surface $\Delta s$ has normal $a_n$ and the line integral is traversed in the direction indicated by the right-hand rule. The vector curl, $\nabla \times A$, is then obtained by combining its three components as given above. The notation “×” is indicative of its vector nature.

We will find the $z$ component of the curl using equation (1.100); the other components follow in a similar fashion. First we compute the line integral indicated in Figure 1–24:

$$\oint A \cdot dl = \int_1^2 A \cdot dl + \int_2^3 A \cdot dl + \int_3^4 A \cdot dl + \int_4^1 A \cdot dl$$

(1.101)
The right-hand rule indicates that we follow the counterclockwise path shown. If the surface is sufficiently small, \( \mathbf{A} \) is approximately constant on each segment, so we have

\[
\oint \mathbf{A} \cdot d\mathbf{l} \approx \mathbf{A}(x, y) \cdot \int_1^2 d\mathbf{l} + \mathbf{A}(x + \Delta x, y) \cdot \int_2^3 d\mathbf{l} + \mathbf{A}(x, y + \Delta y) \cdot \int_4^4 d\mathbf{l} + \mathbf{A}(x, y) \cdot \int_4^1 d\mathbf{l}
\]

(1.102)

The line integrals along the segments are

\[
\int_1^2 d\mathbf{l} = a_x \Delta x \quad \int_2^3 d\mathbf{l} = a_y \Delta y
\]

\[
\int_4^4 d\mathbf{l} = -a_x \Delta x \quad \int_4^1 d\mathbf{l} = -a_y \Delta y
\]

(1.103)

We now use the Taylor series expansion of a function \( f(x, y) \) of two variables, and retain only the low-order terms since \( \Delta s \) will tend to zero:

\[
f(x, y) \approx f(x_0, y_0) + \frac{\partial f(x, y)}{\partial x} \bigg|_{x=x_0, y=y_0} \Delta x + \frac{\partial f(x, y)}{\partial y} \bigg|_{x=x_0, y=y_0} \Delta y
\]

(1.104)

Application to each term in (1.102) gives

\[
\oint \mathbf{A} \cdot d\mathbf{l} \approx [A_x(x, y) ] \Delta x + \left[A_y(x, y) + \frac{\partial A_y(x, y)}{\partial x} \right]_{x=x_0, y=y_0} \Delta x \Delta y
\]

\[
- [A_x(x, y) + \frac{\partial A_x(x, y)}{\partial y} \bigg|_{x=x_0, y=y_0}] \Delta y - [A_y(x, y)] \Delta y
\]

(1.105)

We have found the \( z \) component of the curl; the other components follow in a similar fashion. Collecting all terms in Cartesian coordinates yields the result for \( \nabla \times \mathbf{A} \). It is most easily expressed and remembered as the determinant of a matrix:

\[
\begin{vmatrix}
\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \\
\end{vmatrix}
\]
\[ \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z \] (1.106)

We derived the curl in Cartesian coordinates. The extension to cylindrical and spherical coordinates follows. In cylindrical coordinates, we have

\[ \nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho \mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \mathbf{a}_\rho + \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \mathbf{a}_\phi + \frac{1}{\rho} \left( \frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \mathbf{a}_z \] (1.107)

In spherical coordinates, we have

\[ \nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \begin{vmatrix} \mathbf{a}_r & r \mathbf{a}_\theta & r \sin \theta \mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix} = \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial A_\phi}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right) \mathbf{a}_\theta + \frac{1}{r} \left( \frac{\partial (r A_\theta)}{\partial \phi} - \frac{\partial A_r}{\partial \theta} \right) \mathbf{a}_\phi \] (1.108)

**EXAMPLE 1.17**

Find the curl of the vector field \( \mathbf{A} = \omega \rho e^{-\rho/\alpha^2} \mathbf{a}_\phi \), where \( \alpha \) and \( \omega \) are constant

**Solution:**

From equation (1.107), the curl is given by
\[ \nabla \times \mathbf{A} = \frac{1}{\rho} \begin{bmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho \left( \omega \rho e^{-\left(\frac{\rho}{\alpha}\right)^2} \right) & 0 \end{bmatrix} \]

\[ = \frac{1}{\rho} \begin{bmatrix} (0) a_\rho + (0) \rho a_\phi + \frac{\partial}{\partial \rho} \left( \omega \rho^2 e^{-\left(\frac{\rho}{\alpha}\right)^2} \right) a_z \\ 0 \end{bmatrix} \]

\[ = 2 \omega e^{-\left(\frac{\rho}{\alpha}\right)^2} \left[ 1 - \left( \frac{\rho}{\alpha} \right)^2 \right] a_z \]

**MATLAB Solution:**

We can symbolically work out this example:

```matlab
syms rho phi z omega alpha
A = [0 omega*rho*exp(-(rho/alpha)^2) 0];
curlA = [diff(A(3),phi)/rho-diff(A(2),z)…
         diff(A(1),z)-diff(A(3),rho)  …
         (diff(A(1),phi)-diff(rho*A(2),rho))/rho];
curlA = simplify(curlA)
```

Using MATLAB function `curl`, we can numerically calculate the curl of this field. A plot of the field is shown in (a) and contours of the z-component of the curl are shown in (b). The code of this example is listed in ex117.m.

![MATLAB Figure for Example 1.17](image-url)
From the definition of the curl of a vector given in (1.63), we can obtain Stokes’s Theorem that relates a closed-line integral to a surface integral. Following the same “hand-waving” procedure that we used to derive the divergence theorem, we write

$$\oint A \cdot dl \approx (\nabla \times A)\Delta s \approx \int \int \nabla \times A \cdot ds$$

(1.109)

This is finally written with the same caveats that we employed previously as Stoke’s Theorem.

$$\oint A \cdot dl = \int \int \nabla \times A \cdot ds$$

(1.110)

The line integral on the left-hand side is along the perimeter of the surface in the direction indicated by the right-hand rule. Recall that in the right-hand convention that we are employing, the fingers of the right hand follow the path of the line integral \(dl\) and the thumb points in the direction of the vector surface element \(ds\).

**EXAMPLE 1.18**

Given a vector field \(A = xy a_x - 2x a_y\), verify Stokes’s Theorem over one-quarter of a circle whose radius is 3.

![Figure for Example 1.18](image)

**Solution:**

We must first calculate \(\nabla \times A\) and the surface integral.

\[
\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -(2 + x) a_z
\]

The surface integral becomes
The integral \( \int_{y=0}^{3} \int_{x=0}^{29} (\nabla \times \mathbf{A}) \cdot dx\,dy \) requires the substitution \( y = 3\sin \phi \) and the identity \( \cos^2 \phi = \frac{1}{2} (1 + \cos 2\phi) \) to transform it to the integral \( 9 \int_{0}^{\pi/2} (1 + \cos^2 \phi)\,d\phi \) which can be evaluated. Therefore, we obtain

\[
-\int_{y=0}^{3} \int_{x=0}^{29} \left[ 2\sqrt{9 - y^2} + \frac{(\sqrt{9 - y^2})^2}{2} \right] \,dy = -9 \frac{\pi}{2} + \frac{27}{2} + \frac{27}{6} = -9 \left( 1 + \frac{\pi}{2} \right)
\]

The closed-line integral involves three terms, and using the right-hand convention for the integration sequence, we write

\[
\oint \mathbf{A} \cdot d\mathbf{l} = \int_{x=0}^{3} \int_{y=0}^{29} \mathbf{A} \cdot dx\,a_x + \int_{arc} \mathbf{A} \cdot d\mathbf{l} + \int_{x=0, y=3} \mathbf{A} \cdot dy\,a_z
\]

The two integrals that are along the two axes will contribute zero to the closed integral because the vector \( \mathbf{A} = 0 \) on the axis. The remaining integral becomes

\[
\oint \mathbf{A} \cdot d\mathbf{l} = \int_{arc} \mathbf{A} \cdot d\mathbf{l} = \int_{arc} (xy\,a_x - 2xa_y) \cdot (dx\,a_x + dy\,a_y + dz\,a_z)
\]

\[
= \int (xy\,dx - 2x\,dy)
\]

\[
= \int_{y=0}^{3} x\sqrt{9 - x^2} \,dx - 2\int_{0}^{3} \sqrt{9 - y^2} \,dy = -9 \left( 1 + \frac{\pi}{2} \right)
\]

As we should expect, the two answers are the same.

**MATLAB Solution:**

The MATLAB code (ex118.m) is listed as follows:

```matlab
syms x y z
A = [x*y -2*x 0];
curlA = [diff(A(3),y)-diff(A(2),z) ... 
    diff(A(1),z)-diff(A(3),x) ... 
    diff(A(2),x)-diff(A(1),y)];
func = curlA(3);
surface_integral = int(int(func,x,0,sqrt(9-y^2)),y,0,3)
line_integral = int(subs(A(1),y,sqrt(9-x^2)), x,3,0) + ... 
    int(subs(A(2),x,sqrt(9-y^2)), y,0,3)
```

**1.7.4 Repeated Vector Operations**

Having defined the vector operations of the gradient, the divergence, and the curl, we may be curious about a repeated vector operation, such as the divergence of the curl.
of a vector. There are several methods of approaching this topic. A straightforward rigorous approach would be to perform the vector operations mechanically in order to find the answer. This approach is left for the problems section at the end of this chapter. A second approach that we will pursue here is based on intuitive arguments. The meaning of the various vector operations will likely become more clear as the discussion is presented.

The three vector operations that will be examined are

\[ \nabla \times \nabla = 0 \]  
\[ \nabla \cdot \nabla V = 0 \]  
\[ \nabla \cdot V \nabla V = \nabla^2 V \]  

Other vector identities exist, and a list of useful vector identities is included in Appendix A.

Equation (1.111) can be interpreted in the following terms. The curl operation gives the magnitude and sense of vector rotation confined within a prescribed region. The quantity that this vector represents neither enters nor leaves the region. The divergence operation monitors a vector field's entry into or departure from a region due to a local source or sink within it. Therefore, a vector \( \mathbf{A} \) that has a nonzero curl just rotates and neither enters nor leaves the region. One could think of a boat in a rotating whirlpool that cannot be paddled away from its impending doom as an example of this identity.

Equation (1.112) is understood from the following argument. The gradient of a scalar function expresses the direction and magnitude that an inertialess ball would take as it rolls down a mountain along the path of least resistance. This path would not be expected to close upon itself. The curl, however, would require that the ball return to the same point on the mountain to indicate rotation. This point could be back at the top, interrupting the ball's roll down the mountain under its own volition. Hence we can conclude that (1.75) is correct, since it is clear that the ball could not return unless there were suddenly some new laws of nature—such as anti-gravitational forces.

Equation (1.113) is the definition of the Laplacian operation. It states that there is a vector field \( \nabla V \) where \( V \) is some scalar function. The divergence of this vector field determines whether a source or a sink exists at that point. In Cartesian coordinates, the Laplacian operator is written as

\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]  

As will be seen later, this operation is important for finding the potential distribution caused by a charge distribution.

In cylindrical coordinates, the Laplacian operator is

\[ \nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \]  

In spherical coordinates, the Laplacian operator is
\[ \nabla^2 V = \frac{1}{r^2} \frac{\partial \left( r^2 \frac{\partial V}{\partial r} \right) }{\partial r} + \frac{1}{r^3 \sin \theta} \frac{\partial \left( \sin \theta \frac{\partial V}{\partial \theta} \right) }{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} \] (1.116)
1.8 Summary

- The study of electromagnetic fields makes use of vectors. Vector algebra, integral and differential operations, divergence and Stokes’s theorems have been introduced in this chapter. They will be employed in later to develop the basic laws of electromagnetic theory based on the equations that arise from experimental observations.

- A vector has both magnitude and direction. The direction of the vector can be specified by the dimensionless unit vector \( \mathbf{a}_A \) defined by

\[
\mathbf{a}_A = \frac{\mathbf{A}}{A} = \frac{\mathbf{A}}{|A|}
\]

- Vector addition follows the parallelogram rule. In Cartesian coordinate system, the vector addition follows

\[
\mathbf{A} + \mathbf{B} = (A_x + B_x) \mathbf{a}_x + (A_y + B_y) \mathbf{a}_y + (A_z + B_z) \mathbf{a}_z
\]

- The dot product of two vectors results in a scalar. The definition follows:

\[
\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}| |\mathbf{B}| \cos \theta_{AB} \\
\mathbf{A} \cdot \mathbf{B} \equiv A_x B_x + A_y B_y + A_z B_z
\]

- The vector product or cross-product of two vectors results in another vector and it is defined as

\[
\mathbf{A} \times \mathbf{B} \equiv |\mathbf{A}| |\mathbf{B}| \sin \theta_{AB} \ \mathbf{a}_{A \times B} \\
\hat{\mathbf{a}} \times \mathbf{B} = \begin{vmatrix}
\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix} \\
= (A_y B_z - A_z B_y) \mathbf{a}_x + (A_z B_x - A_x B_z) \mathbf{a}_y + (A_x B_y - A_y B_x) \mathbf{a}_z
\]

- A summary of the unit vectors, the differential lengths, the differential surfaces, and the differential volumes for the three coordinate systems is given in Table 1–6. A summary of the transformations of the variables between coordinate systems is given in Table 1–7.
<table>
<thead>
<tr>
<th>Coordinate system</th>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coordinate variables</td>
<td>$(x, y, z)$</td>
<td>$(\rho, \phi, z)$</td>
<td>$(r, \theta, \phi)$</td>
</tr>
<tr>
<td>Unit vectors</td>
<td>$a_x, a_y, a_z$</td>
<td>$a_\rho, a_\phi, a_z$</td>
<td>$a_r, a_\theta, a_\phi$</td>
</tr>
<tr>
<td>Unit vectors properties</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_x \cdot a_x = a_y \cdot a_y = a_z \cdot a_z = 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_x \cdot a_y = a_y \cdot a_z = a_z \cdot a_x = 0$</td>
<td>$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1$</td>
<td>$a_r \cdot a_r = a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1$</td>
<td></td>
</tr>
<tr>
<td>$a_x \times a_y = a_z$</td>
<td>$a_\rho \times a_\phi = a_z$</td>
<td>$a_r \times a_\theta = a_\phi$</td>
<td></td>
</tr>
<tr>
<td>$a_y \times a_z = a_x$</td>
<td>$a_\phi \times a_z = a_\rho$</td>
<td>$a_\theta \times a_\phi = a_r$</td>
<td></td>
</tr>
<tr>
<td>$a_z \times a_x = a_y$</td>
<td>$a_z \times a_\rho = a_\phi$</td>
<td>$a_\phi \times a_r = a_\theta$</td>
<td></td>
</tr>
<tr>
<td>Differential length $dl$</td>
<td>$dx a_x + dy a_y + dz a_z$</td>
<td>$d\rho a_\rho + \rho d\phi a_\phi + dz a_z$</td>
<td>$dra_r + r d\theta a_\theta + r \sin \theta d\phi a_\phi$</td>
</tr>
<tr>
<td>Differential surface areas</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ds_x = dydz a_x$</td>
<td>$ds_\rho = \rho d\phi dz a_\rho$</td>
<td>$ds_r = r^2 \sin \theta d\theta d\phi a_r$</td>
<td></td>
</tr>
<tr>
<td>$ds_y = dxdz a_y$</td>
<td>$ds_\phi = d\rho dz a_\phi$</td>
<td>$ds_\theta = r dr d\phi a_\theta$</td>
<td></td>
</tr>
<tr>
<td>$ds_z = dxdy a_z$</td>
<td>$ds_z = \rho d\rho d\phi a_z$</td>
<td>$ds_\phi = r dr d\theta a_\phi$</td>
<td></td>
</tr>
<tr>
<td>Differential volume $dv$</td>
<td>$dxdydz$</td>
<td>$\rho d\rho d\phi dz$</td>
<td>$r^2 \sin \theta d\theta d\phi$</td>
</tr>
<tr>
<td>Vector (A) Representation</td>
<td>$A_x a_x + A_y a_y + A_z a_z$</td>
<td>$A_\rho a_\rho + A_\phi a_\phi + A_z a_z$</td>
<td>$A_r a_r + A_\theta a_\theta + A_\phi a_\phi$</td>
</tr>
<tr>
<td>Dot Product $A \cdot B$</td>
<td>$A_x B_x + A_y B_y + A_z B_z$</td>
<td>$A_\rho B_\rho + A_\phi B_\phi + A_z B_z$</td>
<td>$A_r B_r + A_\theta B_\theta + A_\phi B_\phi$</td>
</tr>
<tr>
<td>Dot Product $A \times B$</td>
<td>$a_x a_y a_z$</td>
<td>$a_\rho a_\phi a_z$</td>
<td>$a_r a_\theta a_\phi$</td>
</tr>
</tbody>
</table>
Three integrals are discussed in the chapter:

1. **Line integral of a vector field** $\mathbf{F}$ along a prescribed path from the location $a$ to the location $b$, $\int_a^b \mathbf{F} \cdot d\mathbf{l}$. If the closed line integral $\oint S \mathbf{F} \cdot d\mathbf{s}$ over all possible paths are equal to zero, then the vector $\mathbf{F}$ is called **conservative field**.

2. **Surface integral of a vector field** $\mathbf{A}$ through a surface $S$, $\iint_S \mathbf{A} \cdot d\mathbf{s}$.

3. **Volume integral of a density** $\rho_v$ over the volume $V$, $\iiint_V \rho_v \, dv$.

### TABLE 1–7  Summary of the Transformation between Coordinate Systems

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Coordinate variables</th>
<th>Vector components</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cartesian to cylindrical</strong></td>
<td>$\rho = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1} \left( \frac{y}{x} \right)$, $z = z$</td>
<td>$[A_\rho] = \begin{bmatrix} \cos \phi &amp; \sin \phi &amp; 0 \ -\sin \phi &amp; \cos \phi &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix}$, $A_x$, $A_y$, $A_z$</td>
</tr>
<tr>
<td><strong>Cylindrical to Cartesian</strong></td>
<td>$x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$</td>
<td>$[A_x] = \begin{bmatrix} \cos \phi &amp; -\sin \phi &amp; 0 \ \sin \phi &amp; \cos \phi &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix}$, $A_y$, $A_z$</td>
</tr>
<tr>
<td><strong>Cartesian to spherical</strong></td>
<td>$r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$, $\phi = \tan^{-1} \left( \frac{y}{x} \right)$</td>
<td>$[A_r] = \begin{bmatrix} \sin \theta \cos \phi &amp; \sin \theta \sin \phi &amp; \cos \theta \ \cos \theta \cos \phi &amp; \cos \theta \sin \phi &amp; -\sin \theta \ -\sin \phi &amp; \cos \phi &amp; 0 \end{bmatrix}$, $A_\phi$, $A_\theta$</td>
</tr>
<tr>
<td><strong>Spherical to Cartesian</strong></td>
<td>$x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$</td>
<td>$[A_x] = \begin{bmatrix} \sin \theta \cos \phi &amp; \cos \theta \cos \phi &amp; -\sin \phi \ \sin \theta \sin \phi &amp; \cos \theta \sin \phi &amp; \cos \phi \ \cos \theta &amp; -\sin \theta &amp; 0 \end{bmatrix}$, $A_y$, $A_z$</td>
</tr>
<tr>
<td><strong>Cylindrical to spherical</strong></td>
<td>$r = \sqrt{\rho^2 + z^2}$, $\theta = \cos^{-1} \left( \frac{z}{\sqrt{\rho^2 + z^2}} \right)$, $\phi = \phi$</td>
<td>$[A_r] = \begin{bmatrix} \sin \theta &amp; 0 &amp; \cos \theta \ \cos \theta &amp; 0 &amp; -\sin \theta \ 0 &amp; 1 &amp; 0 \end{bmatrix}$, $A_\phi$, $A_\theta$</td>
</tr>
<tr>
<td><strong>Spherical to Cylindrical</strong></td>
<td>$\rho = r \sin \theta$, $\phi = \phi$, $z = r \cos \theta$</td>
<td>$[A_\rho] = \begin{bmatrix} \sin \theta &amp; \cos \theta &amp; 0 \ 0 &amp; 0 &amp; 1 \ \cos \theta &amp; -\sin \theta &amp; 0 \end{bmatrix}$, $A_\phi$, $A_\theta$</td>
</tr>
</tbody>
</table>
• Three differential operations of fields, the gradient of scalar field, the divergence of vector field, and the curl of vector field are discussed in this chapter. In Cartesian, cylindrical and spherical coordinates, their expressions together with Laplacian operation are summarized in TABLE 1-8.

**TABLE 1–8** Differential Operations in Three Orthogonal Coordinate Systems

<table>
<thead>
<tr>
<th>Coordinate system</th>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient of scalar field $\nabla V$</td>
<td>$\frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z$</td>
<td>$\frac{\partial V}{\partial \rho} \mathbf{a}<em>\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}</em>\phi + \frac{\partial V}{\partial z} \mathbf{a}_z$</td>
<td>$\frac{\partial V}{\partial r} \mathbf{a}<em>r + \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}</em>\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi$</td>
</tr>
<tr>
<td>Divergence of vector field $\nabla \cdot \mathbf{A}$</td>
<td>$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$</td>
<td>$1 \frac{\partial (\rho A_x)}{\rho \partial \rho} + 1 \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$</td>
<td>$1 \frac{\partial (r^2 A_r)}{r^2 \partial r} + \frac{1}{r} \frac{\partial (A_\theta \sin \theta)}{\partial \theta}$</td>
</tr>
<tr>
<td>Curl of vector field $\nabla \times \mathbf{A}$</td>
<td>$\begin{vmatrix} \mathbf{a}_x &amp; \mathbf{a}_y &amp; \mathbf{a}_z \ \frac{\partial}{\partial x} &amp; \frac{\partial}{\partial y} &amp; \frac{\partial}{\partial z} \ A_x &amp; A_y &amp; A_z \end{vmatrix}$</td>
<td>$1 \frac{\partial}{\rho \partial \rho} \begin{vmatrix} \mathbf{a}<em>x &amp; \rho \mathbf{a}</em>\phi &amp; \mathbf{a}_z \ A_x &amp; A_y &amp; A_z \end{vmatrix}$</td>
<td>$1 \frac{\partial}{r \sin \theta \partial \phi} \begin{vmatrix} \mathbf{a}<em>r &amp; r \mathbf{a}</em>\theta &amp; r \sin \theta \mathbf{a}<em>\phi \ A_r &amp; r A</em>\theta &amp; r \sin \theta A_\phi \end{vmatrix}$</td>
</tr>
<tr>
<td>Laplacian of scalar field $\nabla^2 V = \nabla \cdot \nabla V$</td>
<td>$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$</td>
<td>$1 \frac{\partial}{\rho \partial \rho} \begin{vmatrix} \mathbf{a}_x &amp; \mathbf{a}_y &amp; \mathbf{a}_z \end{vmatrix}$</td>
<td>$1 \frac{\partial}{r^2 \partial r} \begin{vmatrix} \mathbf{a}<em>r &amp; r \mathbf{a}</em>\theta &amp; r \sin \theta \mathbf{a}<em>\phi \ A_r &amp; r A</em>\theta &amp; r \sin \theta A_\phi \end{vmatrix}$</td>
</tr>
</tbody>
</table>

**Divergence Theorem**

$$\iiint A \cdot \mathbf{ds} = \iiint (\nabla \cdot \mathbf{A}) \, dv$$

This theorem allows us to move easily between a volume integral and a closed-surface integral.
- **Stokes’s Theorem**

\[ \oint A \cdot dl = \iint_S \nabla \times A \cdot ds \]

The line integral on the left-hand side is along the perimeter of the surface in the direction indicated by the right-hand rule. This theorem relates a closed-line integral to a surface integral.
1.9 Problems

Vector Addition and Subtraction

1.1 Given two vectors \( A = 3a_x + 2a_y \) and \( B = -4a_x + 5a_y \), find
(a) \( |A| \) and \( |B| \)
(b) \( a_A \) and \( a_B \)
(c) \( C = A + B \)
(d) \( D = A - B \).
In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

1.2 Given two vectors \( A = 2a_x \) and \( B = -1a_x + 3a_y \), find
(a) \( C = A + 2B \)
(b) \( D = 3A - B \)
(c) \( |C| \) and \( |D| \)
(d) unit vectors \( a_C \) and \( a_D \)
In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

1.3 Given two vectors \( A = 3a_x + 4a_y + 5a_z \) and \( B = -5a_x + 4a_y - 3a_z \), find
(a) \( |A| \) and \( |B| \)
(b) \( a_A \) and \( a_B \)
(c) \( C = A + B \)
(d) \( D = A - B \).
In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

1.4 Given two vectors \( A = a_x + a_y + a_z \) and \( B = 2a_x + 4a_y + 6a_z \), find
(a) \( C = 3A + B \)
(b) \( D = A - 2B \)
(c) \( |C| \) and \( |D| \)
(d) \( a_C \) and \( a_D \)
In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

Vector Multiplication

1.5 Given two vectors \( A = a_x + a_y + a_z \) and \( B = 2a_x + 3a_y + 6a_z \), find
(a) the scalar product \( A \cdot B \)
(b) the angle between \( A \) and \( B \)
(c) the projection of \( A \) on \( B \) and the projection of \( B \) on \( A \)
(d) the vector product \( A \times B \)
(e) the area of the parallelogram spanned by \( A \) and \( B \)
In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

1.6 Given two vectors \( A = 3a_x + 3a_y + 6a_z \) and \( B = -4a_x + 5a_y - 3a_z \), find
(a) the scalar product $\mathbf{A} \cdot \mathbf{B}$
(b) $\mathbf{C} = \mathbf{A} + 3\mathbf{B}$
(c) $\mathbf{D} = -2\mathbf{A} + \mathbf{B}$
(d) the angle between $\mathbf{A}$ and $\mathbf{C}$
(e) the projection of $\mathbf{C}$ on $\mathbf{D}$

(a) the vector product $\mathbf{C} \times \mathbf{D}$ and prove it is equal to $7(\mathbf{A} \times \mathbf{B})$
(b) the area of the parallelogram spanned by $\mathbf{B}$ and $\mathbf{D}$

In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

1.7 Given two vectors $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y - 7\mathbf{a}_z$, $\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y - 7\mathbf{a}_z$,
(a) find $A_x$ and $A_z$ if $\mathbf{A}$ is parallel to $\mathbf{B}$
(b) find $A_y$ and $A_z$ if $\mathbf{A}$ is perpendicular to $\mathbf{B}$ and the magnitude of $\mathbf{A}$ is 20

1.8 Given two vectors $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z$ and $\mathbf{B} = 3\mathbf{a}_x + 4\mathbf{a}_y - 7\mathbf{a}_z$, find a vector whose magnitude is 11 and perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

1.9 For two vectors $\mathbf{A} = -2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 6\mathbf{a}_x - 5\mathbf{a}_y + 3\mathbf{a}_z$, find
(a) vector $\mathbf{D}$ that is the vector component of $\mathbf{B}$ in the direction of $\mathbf{A}$
(b) vector $\mathbf{G}$ that is the vector component of $\mathbf{B}$ perpendicular to $\mathbf{A}$

1.10 Given three vectors $\mathbf{A} = -2\mathbf{a}_x + 3\mathbf{a}_y + 4\mathbf{a}_z$, $\mathbf{B} = 7\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{C} = -1\mathbf{a}_x + 3\mathbf{a}_y + 5\mathbf{a}_z$, find
(a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$
(b) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(c) the volume of a parallelepiped defined by vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$
(d) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
(e) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

In addition, carefully illustrate these vectors if you solve the problem using MATLAB.

1.11 For vectors $\mathbf{A} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{C} = -1\mathbf{a}_x + 3\mathbf{a}_y + 5\mathbf{a}_z$, show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

Coordinate Systems

1.12 Find the vector $\mathbf{A}$ that connects the two opposite corners of a cube whose volume is $a^3$. One corner of the cube is located at the center of a Cartesian coordinate system. Also write this vector in terms of the magnitude times a unit vector. Find the vector $\mathbf{B}$ from the origin to the opposite corner that lies in the $xoy$ plane.
1.13 For two lines given by \( y = 3x + 6 \) and \( x + 3y = 5 \) in \( xoy \) plane, find
   (1) the unit vectors that are parallel to them
   (2) the smaller angle between them at the intersection point.

1.14 For a line given by \( x - 2y + 8 = 0 \) in \( xoy \) plane, \( Q \) is a point on the line.
   Line \( PQ \) is perpendicular to the line where the Cartesian coordinate of \( P \) is \((2, 2)\).
   Find the coordinate of \( Q \).

1.15 A plane in three dimensional Cartesian system is described as
   \[ 3x + 2y + 5z = 0 \]
   Find a unit vector that is perpendicular to this plane.

1.16 In Cartesian system, a line passing through \( A(-2,4,1) \) and \( B(1,2,3) \) is
   perpendicular to another line passing through \( P(3, -4,2) \) and \( Q \) where \( Q \) is on the line \( AB \).
   Find the coordinate of point \( Q \).

1.17 There are four points \( P(-3,3,2), Q(4, -3,6), G(2,4,5) \) and \( S(2,2,-5) \) in
   Cartesian system. Find
   (1) \( \mathbf{R}_{SP} \), \( \mathbf{R}_{SQ} \) and \( \mathbf{R}_{SG} \)
   (2) the area of triangle \( QGS \)
   (3) the volume of tetrahedral \( PQGS \)

1.18 A vector field \( \mathbf{A} \) is given as \( \mathbf{A}(x, y) = 5x^2y \mathbf{a}_x + 3xy \mathbf{a}_y \).
   Find
   (1) the unit vectors of \( \mathbf{A} \) at \((1, -2)\) and \((2, 3)\)
   (2) plot \( A_x \) versus \( x \) for \( x \) from \(-2\) to \( 2 \) using MATLAB
   (3) plot \( A_x \) versus \( x \) and \( y \) for \(-2 \leq x \leq 2 \) and \(-2 \leq y \leq 2 \) using MATLAB function \( \text{surf} \)
   (4) plot \( \mathbf{A} \) using MATLAB function \( \text{quiver} \) for \(-2 \leq x \leq 2 \) and \(-2 \leq y \leq 2 \)
   (5) draw the contour plot of \( |\mathbf{A}| \) using MATLAB function \( \text{contour} \) for \(-2 \leq x \leq 2 \)
   and \(-2 \leq y \leq 2 \)

1.19 For the following points in Cartesian system, convert their coordinates to
   cylindrical and spherical systems.
   (1) \( A(-2, 1, 0) \)
1.20  For the following points in spherical system \((\rho, \phi, z)\), convert their coordinates to Cartesian and cylindrical systems.
   (1) \(C(3, \frac{5\pi}{4}, -7)\)
   (2) \(D(5, 237^\circ, 8)\)

1.21  For the following points in spherical system \((r, \theta, \phi)\), convert their coordinates to Cartesian and cylindrical systems.
   (1) \(P(2, \frac{5\pi}{6}, \frac{\pi}{4})\)
   (2) \(Q(4, 39^\circ, 70^\circ)\)

1.22  For the points described in Problems 1.19 – 1.21, find
   (1) the distance between \(D\) and \(Q\)
   (2) the area of triangle \(BCQ\)
   (3) the volume of tetrahedral \(ACPQ\)

1.23  Express vector field \(\mathbf{A} = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z\) in cylindrical and spherical systems.

1.24  Express vector field \(\mathbf{F}(x, y, z) = 3za_x - 4y^2za_y + 2\sin x\mathbf{a}_z\) in cylindrical and spherical systems.

1.25  Express vector field \(\mathbf{P} = 3\mathbf{a}_x + 4\mathbf{a}_\rho + 5\mathbf{a}_z\) in Cartesian and spherical systems.

1.26  Express vector field \(\mathbf{G}(\rho, \phi, z) = \rho \cos \phi \mathbf{a}_\rho + z\mathbf{a}_\phi + 2\tan \phi \mathbf{a}_z\) in Cartesian and spherical systems.

1.27  Express vector field \(\mathbf{Q} = 3\mathbf{a}_x + 4\mathbf{a}_\theta + 5\mathbf{a}_\phi\) in Cartesian and cylindrical systems.

1.28  Express vector field \(\mathbf{D}(r, \theta, \phi) = \sin \phi \mathbf{a}_r + ra_\theta - 3r^3 \sin \theta \tan \phi \mathbf{a}_\phi\) in Cartesian and cylindrical systems.

1.29  For vector fields \(\mathbf{P}\) and \(\mathbf{Q}\) described last P1.26 and P1.28, find the angle between them at point \(D(1,2, -3)\) (in Cartesian coordinate). Also find a unit vector that is perpendicular to both of them at \(D\).

**Integral Relations for Vectors**

1.30  Calculate the work required to move a mass \(m\) against a force field \(\mathbf{F} = 5\mathbf{a}_x + 7\mathbf{a}_y\) along the indicated direct path from point \(A\) to point \(B\).
1.31 Calculate the work $W$ required to move a mass against a force field \( \mathbf{F} = 2xy \mathbf{a}_x - 4y^2z \mathbf{a}_y \) along the following path from point $A$ to $B$ as shown in last problem.

1.32 Calculate the work required to move a mass $m$ against a force field $\mathbf{F} = ya \mathbf{a}_x + xa \mathbf{a}_y$ along the path $ABC$ and along the path $ADC$. Is this field conservative?

1.33 Calculate the work required to move a mass $m$ against a force field $\mathbf{F} = \rho \mathbf{a}_\rho + \rho \phi \mathbf{a}_\phi$ along the path $ABC$. 
1.34 Calculate the work required to move a mass \( m \) against a force field \( \mathbf{F} = \rho \phi \mathbf{a}_\phi \) along a closed circle where the radius of the circle is \( a \) and \( 0 \leq \phi \leq 2\pi \). Is this field conservative?

1.35 Find the area of the following surfaces defined by:
   (1) \( \rho = 2, \ 35^\circ \leq \phi \leq 165^\circ, \ 1 \leq z \leq 5 \)
   (2) \( 0 \leq \rho \leq 3, \ \phi = 75^\circ, \ -1 \leq z \leq 7 \)
   (3) \( 2 \leq \rho \leq 7, \ 45^\circ \leq \phi \leq 105^\circ, \ z = 6 \)

1.36 Find the area of the following surfaces defined by
   (1) \( r = 2, \ 25^\circ \leq \theta \leq 105^\circ, \ 15^\circ \leq \phi \leq 93^\circ \)
   (2) \( 2 \leq r \leq 5, \ \theta = 95^\circ, \ 85^\circ \leq \phi \leq 196^\circ \)
   (3) \( 0 \leq r \leq 4, \ 5^\circ \leq \theta \leq 95^\circ, \ \phi \leq 213^\circ \)
1.37 Evaluate the closed-surface integral $\int \int \mathbf{A} \cdot d\mathbf{s}$ if $\mathbf{A} = xa_x + ya_y$ and the surface is that of the cube shown below.

Figure for Problem 1.37

1.38 Evaluate the closed-surface integral of the vector field $\mathbf{A} = xyz a_x + xyz a_y + xyz a_z$ over the cubical surface of last problem. (Hint: Express $\mathbf{A}$ in Cartesian coordinates and then work out the integration.)

1.39 Evaluate the surface integral $\int \int \mathbf{A} \cdot d\mathbf{s}$ if $\mathbf{A} = 3 \rho^2 \cos \phi a_\rho + 5za_z$ and the surface is defined by $\rho = 3, \ 15^\circ \leq \phi \leq 175^\circ, \ 2 \leq z \leq 4$. The normal direction of the surface is $a_\rho$.

1.40 Evaluate the closed-surface integral $\int \int \mathbf{A} \cdot d\mathbf{s}$ if

$\mathbf{A} = 2\rho z^2 \cos \phi a_\rho + 4\rho \sin \phi a_\theta + 5\rho za_z$ and the surface is the outer surface of the volume defined by $2 \leq \rho \leq 7, \ 15^\circ \leq \phi \leq 175^\circ, \ 2 \leq z \leq 4$.

1.41 Evaluate the surface integral $\int \int \mathbf{A} \cdot d\mathbf{s}$ if $\mathbf{A} = -5r^2 \sin \theta \cos \phi a_r$ and the surface is defined by $r = 3, \ 15^\circ \leq \theta \leq 175^\circ, 10^\circ \leq \phi \leq 237^\circ$. The normal direction of the surface is $a_r$.

1.42 Evaluate the closed-surface integral of the vector $\mathbf{A} = 3a_r$ over the spherical surface centered on the origin that has a radius $a$.

1.43 Evaluate the closed-surface integral of the vector $\mathbf{A} = 2ra_r + 4\sin \theta a_\theta + 5r \cos(2\phi)a_\phi$ if the surface is the outer surface of the volume defined by $2 \leq r \leq 6.3, \ 5^\circ \leq \theta \leq 87^\circ, 10^\circ \leq \phi \leq 95^\circ$. 
1.44 Evaluate the closed-surface integral of the vector \( \mathbf{A} = 2 \rho \cos \phi \mathbf{a}_\rho + 3 \sin \phi \mathbf{a}_\phi + 5 z \mathbf{a}_z \), if the surface is the outer surface of the volume defined by \( 1 \leq r \leq 5, \ 45^\circ \leq \theta \leq 90^\circ, 0^\circ \leq \phi \leq 60^\circ \).
(Hint: Find the expression of \( \mathbf{A} \) in spherical coordinate and then evaluate the integral.)

1.45 Find the volume defined by
\[
1 \leq \rho \leq 5, \ 30^\circ \leq \phi \leq 135^\circ, \ 1 \leq z \leq 6
\]
Also find its outer surface area.

1.46 Find the volume defined by
\[
1 \leq r \leq 6, \ 30^\circ \leq \theta \leq 90^\circ, 45^\circ \leq \phi \leq 135^\circ
\]
Also find its outer surface area.

1.47 The mass density is given by \( m = x^3 y z^2 \). Find the total mass within the volume defined by \(-1 \leq x \leq 3, \ -2 \leq y \leq 2, \ 0 \leq z \leq 1\).

1.48 The charge density is given by \( \rho_r = \rho z \cos \phi \). Find the total charge within the volume defined by \( 0 \leq \rho \leq 4, \ 10^\circ \leq \phi \leq 75^\circ, \ -1 \leq z \leq 1\).

1.49 The mass density is given by \( m = r \sin^2 \theta \cos \phi \). Find the total mass within the volume defined by \( 0 \leq r \leq 5, \ 45^\circ \leq \theta \leq 135^\circ, 0^\circ \leq \phi \leq 90^\circ \).

1.50 The charge density is given by \( \rho_r = \rho z \sin(2\phi) \). Find the total charge within the volume defined by \( 0 \leq r \leq 1, \ 30^\circ \leq \theta \leq 90^\circ, 0^\circ \leq \phi \leq 45^\circ \).
(Hint: express the charge density in spherical coordinates and then do volume integration)

**Differential Relations for Vectors**

1.51 A hill can be modeled with the equation \( H = 10 - x^2 - 3y^2 \) where \( H \) is the elevation of the hill. Find the direction in which a frictionless ball would roll unimpeded if released from rest at location \( (x_0, y_0) \).

1.52 Find the gradient of the scalar field \( V = x^3 y z \) and also the directional derivative of \( V \) in the direction specified by the unit vector \( \mathbf{a} = (\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)/\sqrt{3} \) at the point \((1,2,3)\).

1.53 Find the gradient of the following scalar field
\[
(1) \ V = 4xy^2z^2 \\
(2) \ U = 2\rho^2z \cos \phi \\
(3) \ \Phi = r^3 \sin(2\theta) \sin \phi
\]

1.54 By direct differentiation show that
\[
\n\n\n\n\n\n\n\n\]

where

\[R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\]

and \(\nabla'\) denotes differentiation with respect to the variables \(x', y', \text{ and } z'\).

1.55 Find the divergence of the following vector fields

(1) \[\mathbf{A} = x^3 y \sin(\pi z) \mathbf{a}_x + xy \sin(\pi z) \mathbf{a}_y + x^2 y^2 z^2 \mathbf{a}_z\]

(2) \[\mathbf{F} = 2 \rho z^5 \cos \phi \mathbf{a}_\rho + 4 \rho \sin \phi \mathbf{a}_\phi + 5 \rho z \mathbf{a}_z\]

(3) \[\mathbf{G} = 2r \mathbf{a}_r + 4 \sin \theta \mathbf{a}_\theta + 5r \cos(2\phi) \mathbf{a}_\phi\]

Also evaluate their values at the point \((1,1,1)\).

1.56 Show that the divergence theorem is valid for the cube below, located at the center of a Cartesian coordinate system, for a vector \(\mathbf{A} = x \mathbf{a}_x + 2 \mathbf{a}_y\).

1.57 Show that the divergence theorem is valid for a sphere of radius \(a\) located at the center of a coordinate system for a vector field \(\mathbf{A} = r \mathbf{a}_r\).

1.58 The water that flows in a channel with sides at \(x = 0\) and \(x = a\) has a velocity distribution \(v(x, z) = [(a/2)^2 - (x - a/2)^2]^2 \mathbf{a}_y\). The bottom of the river is at \(z = 0\). A small paddle wheel with its axis parallel to the \(z\) axis is inserted into the channel and is free to rotate. Find the relative rates of rotation at the points

\[
\left(x = \frac{a}{2}, z = 1\right), \left(x = \frac{a}{2}, z = 1\right), \text{ and } \left(x = \frac{3a}{4}, z = 1\right)
\]

Will the paddle wheel rotate if its axis is parallel to the \(x\) axis or the \(y\) axis?
1.59 Find curl of the vector fields given in Problem 1.55.

1.60 Evaluate the line integral of the vector function \( \mathbf{A} = x\mathbf{a}_x + x^2\mathbf{a}_y + xyz\mathbf{a}_z \) around the square contour \( C \). Integrate \( \nabla \times \mathbf{A} \) over the surface bounded by \( C \). Show that this example satisfies Stokes’s theorem.

1.61 Show that \( \nabla \times \mathbf{A} = 0 \) if \( \mathbf{A} = 1/\rho \mathbf{a}_\rho \) in cylindrical coordinates.

1.62 Show that \( \nabla \times \mathbf{A} = 0 \) if \( \mathbf{A} = r^2\mathbf{a}_r \) in spherical coordinates.

1.63 In Cartesian coordinates, verify that \( \nabla \cdot \nabla \times \mathbf{A} = 0 \) where \( \mathbf{A} = x^2y^2z^2(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \) by carrying out the indicated derivatives.

1.64 In Cartesian coordinates, verify that \( \nabla \times \nabla \phi = 0 \) where \( \phi = 3x^2y + 4z^2x \) by carrying out the indicated derivatives.

1.65 In Cartesian coordinates, verify that \( \nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A} \) where \( \mathbf{A} = xyz(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \) and \( \phi = 3xy + 4zx \) by carrying out the indicated derivatives.

1.66 In Cartesian coordinates, verify that \( \nabla \cdot (\phi \mathbf{A}) = \mathbf{A} \cdot \nabla \phi + \phi \nabla \cdot \mathbf{A} \) where \( \mathbf{A} = xyz(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \) and \( \phi = 3xy + 4zx \) by carrying out the indicated derivatives.

1.67 By direct differentiation, show that \( \nabla^2(1/R) = 0 \) at all points where \( R \neq 0 \), where

\[
R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}
\]

1.68 Find the Laplacian of the scalar fields given in Problem 1.53.