Part I True/False (15pts)

Let \( p, q, r, s, t \) denote arbitrary statements (propositions). Answer each of the following True/False questions:

**YES** \( \text{NO} \) \( p \land (\neg p \lor q) \) is equivalent to \( p \land q \).
**True.** \( p \land (\neg p \lor q) \equiv (p \land \neg p) \lor (p \land q) \equiv p \land q \).

**YES** \( \text{NO} \) If \( p \Rightarrow q \) is true, then \( (p \land r) \Rightarrow (q \land r) \) is true.
**True.** If \( (p \land r) \) is true, then both \( p \) and \( r \) are true. Thus, both \( q \) and \( r \) are true since \( p \) implies \( q \).

**YES** \( \text{NO} \) If \( p \Rightarrow q \) is false, then \( p \) is false and \( q \) is false.
**False.** When \( (p \Rightarrow q) \) is false, we have \( p \) is true and \( q \) is false.

**YES** \( \text{NO} \) \( (p \lor q) \Rightarrow (s \land t) \Leftrightarrow s \land t \land \neg p \land \neg q \)
**FALSE** De Morgan’s was applied incorrectly.

**YES** \( \text{NO} \) \( (p \land q) \Rightarrow (q \lor r) \) is a tautology.
**TRUE** This is always true, because \( q \) will always imply \( q \) or \( r \).
Part II Shorter Questions (30pts)

1. (20 pts) Prove or disprove these statements for the universe of real numbers (R).
   Partial credit may be given for this question.
   (a) $\forall x \exists y \ [xy = 7]$
   Solution:
   DISPROVE
   Consider $x=0$, there exists no $y$ to make the statement true for that value of $x$.

   (b) $\exists x \forall y \ [2x + 3y = 7]$
   Solution:
   DISPROVE
   The true statement is $\forall x \exists y \ [2x + 3y = 7]$ – no matter what value of $x$ you pick, there will be
   at least one value(in fact, exactly one value) of $y$ to make the statement true.
   BUT, there is no $x$, such that for ALL $y$, the statement will hold. In fact, there is no $x$ for which
   two different values of $y$ exist to make the statement true.

   (c) For the following 2 subproblems, we define $r(x): 2x + 1 = 5$ and $s(x): x^2 = 9$
      1. $\exists x [r(x) \land s(x)]$
      2. $[\exists x r(x) \land \exists x s(x)]$
   Solution:
      1. Disprove. Because there is no one real number $x$ such that $2x + 1 = 5$ and $x^2 = 9$
      2. Prove. There is a real number $b (=2)$ such that $2b + 1 = 5$, and there is a second real number $c$
         (=3) such that $c^2 = 9$
2. (10pts) (From homework assignment) The Rule of the Destructive Dilema is denoted in tabular form as

\[ p \rightarrow q \]
\[ r \rightarrow s \]
\[ \neg q \lor \neg s \]

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\[ \neg p \lor \neg r \]

Determine the related logical implication. Prove the validity of the Rule of the Destructive Dilema by showing that the related logical implication is a tautology (Please ONLY use the elementary laws to simplify the statement).

Solution:

Simply re-interpret the logics and we have \[ r \rightarrow s \iff \neg s \rightarrow \neg r \] and \[ \neg q \lor \neg s \iff q \rightarrow \neg s \]. (Idea: \[ X \rightarrow Y \iff \neg X \lor Y \] and \[ X \rightarrow Y \iff \neg Y \rightarrow \neg X \].)

To prove the tautology, we rearrange the rules as the following:

We get:

\[ p \rightarrow q \quad (1) \]
\[ q \rightarrow \neg s \quad (2) \]
\[ \neg s \rightarrow \neg r \quad (3) \]

--------------- (use Law of Syllogism twice)

\[ p \rightarrow \neg r \iff \neg p \lor \neg r \]

From (1) \rightarrow (2), we get \[ p \rightarrow \neg s \] (Let's call this (2'))

From (2') \rightarrow (3), we get \[ p \rightarrow \neg r \] then we are done.
Part III Longer Questions (30pts) (Justify each step of your proof.)

1. (15 pts.) Let $p$, $q$, $r$ denote arbitrary statements (propositions). Use the truth table method to prove that

$$\neg(p \lor q) \lor r$$

is logically equivalent to

$$(p \rightarrow r) \land (q \rightarrow r).$$

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<th>$q$</th>
<th>$r$</th>
<th>$p \lor q$</th>
<th>$\neg(p \lor q)$</th>
<th>$\neg(p \lor q) \lor r$</th>
<th>$p \rightarrow r$</th>
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<th>$((p \rightarrow r) \land (q \rightarrow r))$</th>
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Note that the column for $\neg(p \lor q) \lor r$ (column 6) and the column for $((p \rightarrow r) \land (q \rightarrow r))$ (column 9) are identical, which proves the equivalence of these two logical expressions.

2. (15pts) Prove that $(q \land (p \rightarrow \neg q)) \rightarrow \neg p$ is a tautology using the laws of logic. (Note: You are not allowed to use truth tables.)

$$(q \land (p \rightarrow \neg q)) \rightarrow \neg p \iff (q \land (\neg p \lor \neg q)) \rightarrow \neg p,$$

Defn of implication

$$\iff (q \land \neg(p \land q)) \rightarrow \neg p,$$

DeMorgan's Law
\[ \iff \neg (q \land \neg(p \land q)) \lor \neg p, \quad \text{Defn of implication} \]

\[ \iff (\neg q \lor \neg (p \land q)) \lor \neg p, \quad \text{DeMorgan's Law} \]

\[ \iff (\neg q \lor (p \land q)) \lor \neg p, \quad \text{Double Negation} \]

\[ \iff (\neg p \lor \neg q) \lor (p \land q), \quad \text{Commutative (or)} \]

\[ \iff \neg(p \land q) \lor (p \land q), \quad \text{DeMorgan's Law} \]

\[ \iff T \quad \text{Inverse Law} \]