Please budget your time well when working on the problems.

Part I True/False (12pts)
Let $A$, $B$, $C$ denote arbitrary finite sets. Answer each of the following True/False questions:

**YES NO** If $A \cup C = B \cup C$, then $A = B$.
**False.** For example, let $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$. Then, $A \cup C = B \cup C = \{1, 2\}$, but $A \neq B$.

**YES NO** If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.
**True.** If $x \in A \cup B$, then $x \in A$ or $x \in B$. In the first case, $x \in C$ because $A \subset C$; similarly, in the second case, when $x \in B$, $x \in C$ is also true because $B \subset C$. Thus, $x \in C$ is true in both cases, which proves $A \cup B \subset C$.

**YES NO** $A \cap B \subset A \times B$.
**False.** The set $A \times B$ contains ordered pairs.

**YES NO** If $|A \cup B| = |A|$, then $|A \cap B| = |B|$.
**True.** By the theorem, $|A \cup B| = |A| + |B| - |A \cap B|$.
Part II Shorter Questions (40pts)

1. (10 pts.) Let \( A \) and \( B \) denote two sets. Prove \( A - B = A - (A \cap B) \). Justify your steps.

\[
A - (A \cap B) \\
= A \cap \neg (A \cap B), \text{ using the theorem } X - Y = X \cap \neg Y \\
= A \cap (\neg A \cup \neg B), \text{ by De Morgan’s law} \\
= (A \cap \neg A) \cup (A \cap \neg B), \text{ Distributive law} \\
= \emptyset \cup (A \cap \neg B), \text{ by the law } A \cap \neg A = \emptyset \\
= A - B, \text{ by the law } \emptyset \cup X = X.
\]

2. (10 pts.) Let \( a, b \) denote two integers. If \( a + b \) is odd, then prove \( a^2 + ab \) is odd or \( b^2 + ab \) is odd.

**Solution one**  
Since \( a + b \) is odd, exactly one of the integers \( a \) and \( b \) is odd, the other is even. Thus, there are two cases:

(Case 1) \( a \) is odd and \( b \) is even. In this case, \( a^2 + ab = a(a + b) \) is odd since both \( a \) and \( a + b \) are odd and the product of two odd integers is odd.

(Case 2) \( b \) is odd and \( a \) is even. In this case, \( b^2 + ab = b(a + b) \) is odd since both \( b \) and \( a + b \) are odd and the product of two odd integers is odd.

Therefore, we proved that either \( a^2 + ab \) is odd (in case 1) or \( b^2 + ab \) is odd (in case 2).

**Solution two**  
We prove the contrapositive. That is, we assume both \( a^2 + ab \) and \( b^2 + ab \) are even ---- (1), then we prove \( a + b \) is even ---- (2). From (1), the difference \( a^2 + ab - (b^2 + ab) = a^2 - b^2 = (a - b)(a + b) \) is even ---- (3). Therefore, (3) implies either \( (a - b) \) is even or \( (a + b) \) is even. In the first case, since \( (a - b) + 2b = a + b \), so \( (a + b) \) is also even (because the sum of two even numbers is even). Thus, \( (a + b) \) is even in both cases, which proves (2).

**Solution three**  
We prove the contrapositive. That is, we assume both \( a^2 + ab \) and \( b^2 + ab \) are even ---- (1), then we prove \( a + b \) is even ---- (2). From (1), the sum \( (a^2 + ab) + (b^2 + ab) = a^2 + 2ab + b^2 = (a + b)(a + b) \) is even. Thus, \( (a + b) \) is even, which proves (2).

3. (10 pts) Prove or disprove: \( A \subseteq B \cap C \Rightarrow (A \cap B = \emptyset) \vee (A \cap C = \emptyset) \)

Solution: DISPROVE

Consider this counter example: \( A = \{1,2\}, B = \{1\}, C = \{2\} \)

The LHS is true in this situation, but the right hand side is false, disproving the assertion.
4. (10 pts) Let $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$. (If you show work, partial credit may be given.)
You may express your answer in terms of powers of 2, and combinations.

a. How many subsets of $A$ of size 3 have the product of their two smallest elements equal to
the largest element? (Note: A set with this property is $\{4, 6, 24\}$.)
ANSWER : 0

b. How many subsets of $A$ contain at least one odd number as an element?
ANSWER : 254

c. How many subsets of $A$ contain no even numbers as elements?
ANSWER : 128

d. How many subsets of $A$ have 4 elements in them?
ANSWER : 70

e. Let $\text{Sum}(B)$ denote the sum of the elements of a set $B$, where $B \subseteq A$. How many distinct
subsets $B$ are there such that $\text{Sum}(B) < 11$?
ANSWER : 11
Part III Longer Questions (23pts) (Justify each step of your proof.)

1. (10 pts.) There are two parts in this question:
   (a) Let $c$ and $d$ denote integers. Suppose $3 \mid c$ is false and $3 \mid d$ is false (that is, suppose 3 is not a divisor of $c$ and 3 is not a divisor of $d$). Prove $3 \mid cd$ is false. (Hint: First explain why $c = 3n + 1$ or $c = 3n + 2$ for some integer $n$.)

   Solution:
   Since $3 \mid c$ is false, so when dividing $c$ by 3, the remainder must be either 1 or 2, by Euclid’s Division Theorem. Thus, $c = 3n + 1$ or $c = 3n + 2$ for some integer $n$. Similarly, since $3 \mid d$ is false, $d = 3k + 1$ or $d = 3k + 2$ for some integer $k$. Therefore, there are four cases to consider:
   (Case 1) Suppose $c = 3n + 1$ and $d = 3k + 1$. In this case, $cd = 9nk + 3n + 3k + 1 = 3(3nk + n + k) + 1$, so $3 \mid cd$ is false.
   (Case 2) Suppose $c = 3n + 1$ and $d = 3k + 2$. In this case, $cd = 9nk + 6n + 3k + 2 = 3(3nk + 2n + k) + 2$, so $3 \mid cd$ is false.
   (Case 3) Suppose $c = 3n + 2$ and $d = 3k + 1$. In this case, $cd = 9nk + 3n + 6k + 2 = 3(3nk + n + 2k) + 2$, so $3 \mid cd$ is false.
   (Case 4) Suppose $c = 3n + 2$ and $d = 3k + 2$. In this case, $cd = 9nk + 6n + 6k + 4 = 3(3nk + 2n + 2k + 1) + 1$, so $3 \mid cd$ is false.

   Therefore, we proved that $3 \mid cd$ is false is all cases.

   (b) Give an example of integers $c$ and $d$ that shows that the following statement is false:
   If $6 \mid c$ is false and $6 \mid d$ is false, then $6 \mid cd$ is false.

   Solution:
   A Counter Example: Let $c = 2$, $d = 3$. Then both $6 \mid c$ is false and $6 \mid d$ is false, but since $cd = 6$, $6 \mid cd$ is true.

2. (13 pts.) Let $A$, $B$, and $C$ denote sets. There are two parts in this question:
   (a) (8pts) Prove if $B - A \subseteq C - A$, then $B \cup A \subseteq C \cup A$.

   Proof:
   We need to prove if $x \in B \cup A$ --- (1), then $x \in C \cup A$ --- (2).
   From (1), $x \in B$ or $x \in A$, so we consider two cases:

   (Case 1) Suppose $x \in A$. In this case, $x \in C \cup A$ because $A \subseteq C \cup A$, so (2) is proved.
(Case 2) Suppose \( x \in B \) --- (3). We may assume \( x \notin A \) --- (4) because the case of \( x \in A \) has been considered in Case 1. Combining (3) and (4) yields \( x \in B - A \) --- (5) by the definition of set difference. Since \( B - A \subseteq C - A \) by assumption, so combining with (5) yields \( x \in C \cup A \), which proves (2).

Therefore, we proved (2) in both cases.

(b) (5pts) Give a small example (of sets \( A, B, \) and \( C \)) that demonstrates that \( B - A \subseteq C - A \) is true but \( B \subseteq C \) is false.

\textbf{Solution:}
Let \( A = \{1\}, B = \{1, 2\}, \) and \( C = \{2\}. \) Then \( B - A = \{2\}, C - A = \{2\}, \) so \( B - A \subseteq C - A \) is true. However, \( B \subseteq C \) is false.

\textbf{Bonus: (5pts)}
What is the instructor's name? Write in the order of first name then last name.

\textbf{Solution:} Pawel Wocjan