Use **induction** to prove each of the following questions, and be sure to **mark clearly when and where the induction hypothesis is applied in each question:**

1. (15pts)
   Prove that $421 \mid (20^n + 21^{2^n} + 21^n)$ for all integer $n \geq 0$.

2. (15pts)
   Suppose a sequence $a_0, a_1, \ldots, a_n, \ldots$ is defined by the following recurrence:
   
   \[ a_0 = 6, \quad a_1 = 13, \quad \text{and} \quad a_n = a_{n-1} + 6a_{n-2}, \quad \text{for} \quad n \geq 2. \]

   Prove that the sequence $a_n$ satisfies the formula $a_n = 5 \cdot (3)^n + (-2)^n$ for all integers $n \geq 0$.

3. (15pts)
   Prove that for integer $n \geq 0$, \( \sum_{j=0}^{n} C(r+j, j) = C(r+n+1, n) \) where $r$ is an arbitrary positive constant (Hint: Recall the Pascal's Triangle identity $C(r-1, k) + C(r-1, k-1) = C(r, k)$.)

4. (15pts)
   Prove that for integer $n \geq 1$, \( \sum_{j=1}^{n} j^5 = \frac{5^{n+1}(4n-1) + 5}{16} \).