Use induction to prove each of the following questions, and be sure to mark clearly when and where the induction hypothesis is applied in each question:

1. (15pts) Prove that $421 \mid (20^{n+2} + 21^{2n+1})$ for all integer $n \geq 0$.

   **Answer:** We use induction on $n \geq 0$.

   (Basis Step) Consider $n = 0$. In this case, $20^{n+2} + 21^{2n+1} = 20^2 + 21 = 421$, so it is divisible by 421, and the Basis Step is proved.

   (Induction Hypothesis) Consider $n = k$. Suppose $421 \mid (20^{k+2} + 21^{2k+1})$ for some integer $k \geq 0$; that is, suppose $20^{k+2} + 21^{2k+1} = 421m$ for some integer $m$.

   (Induction Step) $n = k + 1$. We need to prove $20^{k+1+2} + 21^{2(k+1)+1} = 20^{k+3} + 21^{2k+3}$ is divisible by 421. Note that

   $$20^{k+3} + 21^{2k+3} = 20 \cdot (20^{k+2} + 21^{2k+1}) - 20 \cdot 21^{2k+1} + 21^{2k+3} \quad \text{(by “algebra”)}
   = 20 \cdot 421m + 21^{2k+3} \cdot (20 + 21^2), \text{ by the Induction Hypothesis (and factoring)}
   = 20 \cdot 421m + 21^{2k+3} \cdot 421
   = 421 \cdot (20m + 21^{2k+1}), \text{ which is divisible by 421.}
   $$

   Thus, the Induction Step is proved.

   By induction we proved $421 \mid (20^{n+2} + 21^{2n+1})$ for all integer $n \geq 0$. 
2. (15 pts.) Suppose a sequence $a_0, a_1, \ldots, a_n, \ldots$ is defined by the following recurrence:

$$a_0 = 6, \ a_1 = 13, \text{ and } a_n = a_{n-1} + 6a_{n-2}, \text{ for } n \geq 2.$$  

Prove that the sequence $a_n$ satisfies the formula $a_n = 5 \cdot (3)^n + (-2)^n$ for all integer $n \geq 0$.

**Answer:** We use strong induction on $n \geq 0$.

**(Basis Step)**

When $n = 0$, the formula for $a_n = 5 \cdot (3)^n + (-2)^n = 5 + 1 = 6$, which is equal to the given initial value of $a_0$.

When $n = 1$, the formula for $a_n = 5 \cdot (3)^n + (-2)^n = 5 \cdot 3 + (-2) = 15 - 2 = 13$, which is equal to the given initial value of $a_1$.

Thus, the basis Step is proved for both $n = 0$ and $n = 1$.

**(Induction Hypothesis)**

Suppose $a_n = 5 \cdot (3)^n + (-2)^n$ for all integer $n$ in the range $0 \leq n \leq k$ for some $k \geq 1$.

**(Induction Step)** Consider $n = k + 1$.

We need to prove $a_{k+1} = 5 \cdot (3)^{k+1} + (-2)^{k+1}$ --- (1).

Note that $a_{k+1} = a_k + 6 \ a_{k-1}$, using the recurrence since $k + 1 \geq 2$

$$= (5 \cdot (3)^k + (-2)^k) + 6 \cdot (5 \cdot (3)^{k-1} + (-2)^{k-1}) \text{ by the Induction Hypothesis applied to } n = k \text{ and to } n = k - 1$$

$$= 5 \cdot (3)^k + 6 \cdot 5 \cdot (3)^{k-1} + (-2)^k + 6 \cdot (-2)^{k-1} \text{ by rearranging terms}$$

$$= (3)^{k-1} (5 \cdot 3 + 6 \cdot 5) + (-2)^k \cdot (-2)^{k-1} \text{ by factoring}$$

$$= (3)^{k-1} \cdot (45) + (-2)^{k-1} \cdot (4)$$

$$= (3)^{k-1} \cdot (3^2 \cdot 5) + (-2)^{k-1} \cdot (2)^2$$

$$= 5 \cdot (3)^{k+1} + (-2)^{k+1} = \text{RHS of (1)}$$

Thus, the Induction Step is proved.

By induction we proved the formula $a_n = 5 \cdot (3)^n + (-2)^n$ for all integer $n \geq 0$. 
3. (15pts) Prove that for integer \( n \geq 0 \), \( \sum_{j=0}^{n} C(r + j, j) = C(r + n + 1, n) \) where \( r \) is an arbitrary positive constant (Hint: Recall the Pascal's Triangle identity \( C(r-1, k) + C(r-1, k-1) = C(r, k) \)).

**Answer:** We use induction on \( n \geq 0 \).

*(Basis Step)*

Consider \( n = 0 \). In this case, the LHS = \( \sum_{j=0}^{0} C(r + j, j) = C(r, 0) = 1 \).

The RHS = \( C(r + 0 + 1, 0) = C(r + 1, 0) = 1 \). Thus, LHS = RHS, so the Basis Step is proved.

*(Induction Hypothesis)*

Consider \( n = k \). Suppose \( \sum_{j=0}^{k} C(r + j, j) = C(r + k + 1, k) \) for some \( k \geq 0 \).

*(Induction Step) (when \( n = k + 1 \))*

Need to show \( \sum_{j=0}^{k+1} C(r + j, j) = C(r + k + 1 + 1, k + 1) = C(r + k + 2, k + 1) \) -- (1).

Note that the LHS of (1) = \( \sum_{j=0}^{k} C(r + j, j) + C(r + k + 1, k + 1) \), by the definition of summation

\[
= C(r + k + 1, k) + C(r + k + 1, k + 1), \text{ by the Induction Hypothesis}
\]

\[
= C(r + k + 2, k + 1), \text{ by Pascal's Triangle Identity}
\]

Thus, the Induction Step is proved.

By induction we proved \( \sum_{j=0}^{n} C(r + j, j) = C(r + n + 1, n) \) for all \( n \geq 0 \).
4. (15 pts.) Prove that for integer $n \geq 1$, \( \sum_{j=1}^{n} j5^j = \frac{5^{n+1}(4n-1)+5}{16} \).

**Answer:** We use induction on $n \geq 1$.

**(Basis Step)**

Consider $n = 1$. In this case, the LHS $= \sum_{j=1}^{1} j5^j = 1 \cdot 5^1 = 5$. The RHS $= \frac{5^{1+1}(4 \cdot 1 - 1)+5}{16} = \frac{5 \cdot 5 + 5}{16} = \frac{30}{16} = 5$. Thus, LHS = RHS, so the Basis Step is proved.

**(Induction Hypothesis)** Consider $n = k$.

Suppose $\sum_{j=1}^{k} j5^j = \frac{5^{k+1}(4k - 1)+5}{16}$ for some $k \geq 1$.

**(Induction Step)** Consider $n = k + 1$.

We need to prove $\sum_{j=1}^{k+1} j5^j = \frac{5^{k+2}(4k + 3) + 5}{16}$.

Note that the LHS of (1) $= \sum_{j=1}^{k} j5^j + (k + 1)5^{k+1}$, by the definition of summation.

\[
\begin{align*}
\sum_{j=1}^{k+1} j5^j & = \frac{5^{k+1}(4k - 1)+5}{16} + (k + 1)5^{k+1}, \text{ by the Induction Hypothesis} \\
& = \frac{5^{k+1}(4k - 1)+5 + 16(k + 1)5^{k+1}}{16} \\
& = \frac{5^{k+1}(4k - 1 + 16k + 16) + 5}{16} \\
& = \frac{5^{k+1}(20k + 15) + 5}{16} = \frac{5^{k+1} \cdot 5(4k + 3) + 5}{16} = \frac{5^{k+2}(4k + 3) + 5}{16}
\end{align*}
\]

Thus, the Induction Step is proved.

By induction we proved $\sum_{j=1}^{n} j5^j = \frac{5^{n+1}(4n - 1)+5}{16}$ for all $n \geq 1$. 