Problem #1: (20pts)
Consider the following relation $R$ defined over the set of positive integers:

$$R = \{(x,y) \mid x/y = 4 \lor y/x = 4\}$$

Determine if the relation $R$ is

(i) reflexive, (ii) irreflexive, (iii) symmetric, (iv) anti-symmetric, and (v) transitive.

**Answer:**
(i) No, because $(1,1) \notin R$.
(ii) Yes, it is impossible for $(a,a) \in R$ since we know that $a/a \neq 4$.
(iii) Yes, this relation is symmetric. We must prove:
if $(a,b) \in R$, then $(b,a) \in R$. If $(a,b) \in R$, then we know that either $a/b = 4$, or $b/a = 4$.
In the first case we have $a/b = 4$ which means $b/a = 1/4$, but we will still have $(b,a) \in R$, since $a/b = 4$.
In the second case we have $b/a = 4$, which means that $a/b = 1/4$, but we will still have $(b,a) \in R$, since $b/a = 4$.
(iv) The relation is NOT anti-symmetric since both $(1,4) \in R$ and $(4,1) \in R$.
(v) The relation is NOT transitive. We have $(1,4) \in R$, $(4,1) \in R$, but $(1,1) \notin R$.

Problem #2: (10pts)
Prove or disprove: Let $R$ be a relation over $A \times A$. If $R \circ R$ is transitive, then $R$ is also transitive.

**Answer:**
This is false. Let $R$ be defined over $A \times A$, where $A = \{1,2,3\}$. Let $R = \{(1,2), (2,3)\}$. Now, we have that $R \circ R = \{(1,3)\}$, which is transitive, while $R$ is not.
**Problem 3:** (15pts)

Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) denote two functions. If both \( f \) and \( g \) are surjective, prove that the composition \( g \circ f : A \rightarrow C \) is a surjection as well.

**Answer:**

To prove \( g \circ f \) is surjective, we need to prove that for all elements \( c \in C \), there exists an element \( a \) such that \( g \circ f (a) = c \).

Let \( c \) be an arbitrary element from the set \( C \).
Since \( g \) is surjective, we know there exists an element \( b \) such that \( g(b) = c \).
Similarly, for this element \( b \), since \( f \) is surjective, there exists an element \( a \) such that \( f(a) = b \).
Substituting, we have \( g(f(a)) = c \).
Thus, we have shown that for an arbitrarily chosen element \( c \), we can always find an element \( a \) such that \( g \circ f (a) = c \).

**Problem 4:** (15pts)

Let \( g : A \rightarrow A \) be a bijection. For \( n \geq 2 \), define \( g^n = g \circ g \circ \ldots \circ g \), where \( g \) is composed with itself \( n \) times. Prove that for \( n \geq 2 \), that \( (g^n)^{-1} = (g^{-1})^n \).

**Answer:**

**Base case:** \( n=2 \). LHS = \((g^2)^{-1} = (g \circ g)^{-1} = g^{-1} \circ g^{-1} = g^2 \)

RHS = \((g^{-1})^2 = g^2 \), Hence, RHS = LHS when \( n = 2 \)

**Inductive hypothesis:** Assume for an arbitrary value of \( n=k \) that \((g^k)^{-1} = (g^{-1})^k \).

**Inductive step:** Under this assumption we must show that \((g^{k+1})^{-1} = (g^{-1})^{k+1} \).

\[
(g^{k+1})^{-1} = (g \circ g^k)^{-1}
\]

\[= (g^k)^{-1} \circ g^{-1}, \text{ by the rule for the inverse of a function composition.} \]

\[= (g^{-1})^k \circ g^{-1}, \text{ by the inductive hypothesis} \]

\[= (g^{-1})^{k+1}, \text{ by the definition of function composition.} \]

This proves the inductive step so we can conclude that \((g^n)^{-1} = (g^{-1})^n \) is true for all \( n \geq 2 \).