Problem # 1 (15pts)
The symmetric difference of two sets is defined as $A \oplus B = (A - B) \cup (B - A)$.
Prove that $A \oplus B = (A \cup B) - (A \cap B)$.

Solution:

$A \oplus B = (A - B) \cup (B - A)$

$\iff \{ x | x \in (A - B) \lor x \in (B - A) \}$ by def of the set union

$\iff \{ x | (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \}$ by def of the set difference

$\iff \{ x | ((x \in A \land x \notin B) \lor x \in B) \land ((x \in A \land x \notin B) \lor x \notin A) \}$ by distributive property

$\iff \{ x | ((x \in A \lor x \in B) \land (x \notin B \lor x \notin A)) \land ((x \in A \lor x \notin A) \text{ and } (x \notin B \lor x \notin A)) \}$

by distributive property for logical or , and.

$\iff \{ x | ((x \in A \lor x \in B) \land T)) \land (T \land (x \notin B \lor x \notin A)) \}$, by logical identity $(p \lor \neg p) \equiv T$.

$\iff \{ x | (x \in A \lor x \in B) \land (x \notin B \lor x \notin A) \}$, by logical identity $p$ and $T \equiv p$

$\iff \{ x | x \in A \cup B \land \neg(x \in B \land x \in A) \}$, by def of a set union and Morgan's law

$\iff \{ x | x \in A \cup B \land \neg(x \in A \cap B) \}$, by def of intersection

$\iff \{ x | x \in (A \cup B) - (A \cap B) \}$, by def of set difference

$\iff (A \cup B) - (A \cap B)$

Or you can simply do the following by using def of symmetric difference, def of complement, distributive law numerous times

$$
\begin{align*}
A \oplus B & = (A - B) \cup (B - A) \\
& = [(A \cap \overline{B}) \cup (B \cap \overline{A})] \\
& = [(A \cap \overline{B}) \cup B] \cap [(A \cap \overline{B}) \cup \overline{A}] \\
& = [(A \cup B) \cap (B \cup \overline{B})] \cap [(A \cup \overline{A}) \cap (\overline{A} \cup \overline{B})] \\
& = (A \cup B) \cap (A \cap \overline{B}) \\
& = (A \cup B) - (A \cap B)
\end{align*}
$$
Problem # 2 (15pts)
Let $A$ and $B$ be arbitrary sets. Prove or disprove the following propositions:

a) $\text{Power}(A - B) = \text{Power}(A) - \text{Power}(B)$

b) $\text{Power}(A \cap B) = \text{Power}(A) \cap \text{Power}(B)$

**Solution:**

a) 5pts  

Disprove
Simply let $A = \emptyset$ and $B = \emptyset$.
For this example we have $\emptyset \in \text{Power}(A - B)$ but $\emptyset \notin \text{Power}(A) - \text{Power}(B)$ as $\text{Power}(A) - \text{Power}(B) = \{\emptyset\} - \{\emptyset\} = \emptyset$.

b) 10pts (5pts for each direction)

Need to show i) $\text{Power}(A) \cap \text{Power}(B) \subseteq \text{Power}(A \cap B)$ and ii) $\text{Power}(A \cap B) \subseteq \text{Power}(A) \cap \text{Power}(B)$

**Part I.**  
$\text{Power}(A) \cap \text{Power}(B) \subseteq \text{Power}(A \cap B)$

Let $X$ be an arbitrary of $\text{Power}(A ) \cap \text{Power}(B)$, i.e., $X \in \text{Power}(A) \cap \text{Power}(B)$.

We need to prove that $X \in \text{Power}(A \cap B)$. By definition of intersection, the statement $X \in \text{Power}(A) \cap \text{Power}(B)$ is equivalent to $X \in \text{Power}(A) \land X \in \text{Power}(B)$.

By the definition of a power set this statement is equivalent to $X \subseteq A$ and $X \subseteq B$. By the def of subset this means that for any $x$, we have the logical equivalences

$(x \in X \Rightarrow x \in A \land x \in B) \iff (x \in X \Rightarrow x \in (A \cap B)) \iff X \subseteq A \cap B$

By the definition of a power set we have $X \subseteq A \cap B \iff X \in \text{Power}(A \cap B)$.

**Part II.**  
$\text{Power}(A \cap B) \subseteq \text{Power}(A) \cap \text{Power}(B)$

Take arbitrary element $X$ of $\text{Power}(A \cap B)$. We need to prove that $X \in \text{Power}(A) \cap \text{Power}(B)$. By the definition of a power set $X \in \text{Power}(A \cap B) \iff X \subseteq A \cap B$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $X \subseteq A \cap B \Rightarrow (X \subseteq A \land X \subseteq B)$. By the definition of a power set it means, that $X \in \text{Power}(A) \land X \in \text{Power}(B)$. By the definition of intersection the latter implies the desired statement $X \in \text{Power}(A) \cap \text{Power}(B)$.
Problem # 3 (15pts)
For $U = \mathbb{Z}^+$ (the set of positive integer) let $A \subseteq U$ where $A = \{1, 2, 3, 4, 5, 8, 9, 10, 11, 16, 19, 20\}$. 

a) How many subsets of $A$ contain six elements?

b) How many six-element subsets of $A$ contain four even integers and two odd integers?

c) How many subsets of $A$ contain only odd integers?

Solution: (5pts each)

a) This is the number of combinations of size 6 from 12, $C(12, 6) = \frac{12!}{6! \cdot 6!} = 924$.

b) We can choose four even integers from 6 available $\{2, 4, 8, 10, 16, 20\}$ in $C(6, 4) = \frac{6!}{4! \cdot 2!} = 15$ different ways. Two odd integers can be picked from six available in $A$ in $C(6, 2) = 15$ different ways. Hence, we get $15 \cdot 15 = 225$.

c) That is the number of subsets of a six-element set $2^6 - 1 = 64 - 1 = 63$.

Problem # 4 (20pts)
Let $A$ and $B$ be arbitrary sets. Prove that

a) $(A \cup B) = (A \cap B)$ if and only if $A = B$

b) $A \cap (B - A) = \emptyset$

c) $A \cup (B - A) = A \cup B$

Solution: (a) 10 pts b) 5pts c) 5pts

a) $(A \cup B) = (A \cap B)$ iff $A = B$ means you have to show that $(A \cup B) = (A \cap B) \rightarrow A = B$ and $A = B \rightarrow (A \cup B) = (A \cap B)$.

Part 1. $(A \cup B) = (A \cap B) \rightarrow A = B$

Assume $(A \cup B) = (A \cap B)$ to prove that $A = B$. To prove $A = B$ we can prove two subset relations, $A \subseteq B$ and $B \subseteq A$.

To prove $A \subseteq B$ take arbitrary element $x \in A$ which also means that $x \in A \cup B$. But since $(A \cup B) = (A \cap B)$, $x \in A \cup B$ implies that $x \in A \cap B$. By def. we know if $x \in A \cap B$, then $x \in A$ and $x \in B$. Hence, we show that if $x \in A$ then $(A \cup B) = (A \cap B)$, we know $x \in B$. So, we proved that $A \subseteq B$.

In the same way we can prove that if $(A \cup B) = (A \cap B)$ then $B \subseteq A$. That is equivalent $A = B$.

Part 2. $A = B \rightarrow (A \cup B) = (A \cap B)$.

Assume $A = B$ to prove $(A \cup B) = (A \cap B)$.

$(A \cup B) = (A \cup A)$ (assumption)
$= A$ (idempotent law)
$= (A \cap A)$ (idempotent law)
$= (A \cap B)$ (assumption).

b) $A \cap (B - A) = \emptyset$
We must show that there does NOT exist an element x such that \( x \in A \cap (B - A) \).

If such an element exists, we know that \( x \in A \), by the definition of \( \cap \). Now, we must show that in this situation, \( x \notin B - A \). We know that for \( x \in B - A \), by definition, we must have \( x \in B \) and \( x \notin A \). But we know that this can not be true, since \( x \in A \). Thus, we can conclude that \( x \notin B - A \). This proves that there does NOT exist an element in the set \( A \cap (B - A) \). We can conclude that the set is the empty set.

Or you can simply argue

\[
\begin{align*}
A \cap (B - A) & \iff A \cap (B \cap \overline{A}) \\
& \iff A \cap \overline{A} \cap B \\
& \iff B \cap \emptyset \\
& \iff \emptyset
\end{align*}
\]

c) \( A \cup (B - A) \)

\[
\begin{align*}
& \iff A \cup (B \cap \overline{A}) \\
& \iff (A \cup B) \cap (A \cup \overline{A}) \\
& \iff (A \cup B) \cap U \\
& \iff (A \cup B)
\end{align*}
\]