Honors Introduction to Discrete Structures

COT 3100H

Term: Spring 2007
Time: Tues Thurs 1:30pm - 2:45pm
Location: ENGR 0227

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Office hours: Wed 3:00pm – 4:15pm, HC 365
Lecture Notes:
The lectures notes, and other material used during the semester such as the assignments, exams, their solution keys, will be available and updated on the web at


Text:

Course Objectives:
This course provides an introduction to discrete mathematics that is relevant to future computer science courses:

- Logic
- Set Theory
- Proof Methods
- Principles of Counting
- Basic Number Theory: Properties of Integers; Mathematical Induction
- Functions and Relations
- ... let's see how much time is left
Grading Policy:

- **Homework Assignments: 25%**
  every two weeks or so; you have one week to work on the assignments

- **First Midterm Exam: 25%**
  February 20\(^{th}\) and February 22\(^{nd}\)

- **Second Midterm Exam: 25%**
  March 22\(^{nd}\)

- **Final Exam (see University's Final Exam Schedule): 25%**

Grades A, B, C, D, and F are based on the straight-percentage scale (that is, 90% or above is A, 80% to 89.99% is B, etc.)

**Note:**
All work (homework assignments and exams) represent individual effort; no late homework and make-up tests will be accepted except for exceptional situations (handled individually); all exams are closed books, closed notes, and no calculators
Academic Integrity and Student Conduct:
Please read and understand student rights and responsibilities including conduct rules clearly stated in UCF’s golden rules, at

http://www.goldenrule.sdes.ucf.edu/2e_Rules.html
**Logic**

- we take a close look at what constitutes a valid argument and a mathematical proof
- when a mathematician wishes to provide a proof for a given situation, he or she must use a system of logic
- this is also true when a computer scientist develops and implements an algorithms needed to solve a certain problem
- the logic of mathematics is applied to decide whether one statement follows from, or is a logical consequence of, one or more other statements
Logic

- Propositional Calculus
  - Propositions
  - Propositional Connectives
  - Truth Tables
  - Formulas in Propositional Calculus
  - Truth Functions
  - Logical Equivalence: The Laws of Logic
  - Logical Implication: Rules of Inference
  - Tautologies, Inferences in Mathematics

- Formula in Predicate Calculus
  - Predicates
  - Quantifiers
  - Formulas in Predicate Calculus
  - Interpretation of Predicate Calculus Formulas
  - Logically Valid Formulas
  - The Rule of Universal Specification
  - The Rule of Universal Generalization
  - Restricted Quantification
Propositions

- a proposition (or statement) is the mental reflection of a fact, expressed as a statement in a natural or artificial language
- every proposition is considered to be true or false
- this is the principle of two-valuedness (in contrast to many-valued or fuzzy logic)
- “true” and “false” are called the truth value of the proposition and they are denoted by 1 and 0, respectively
- the truth values are considered as propositional constants
- we use lower case letters such as $p$, $q$ and $r$ to represent propositions
Example:

\[ p: \text{COT3100H is a required course} \]
\[ q: \text{Students must pass the Foundation Exam} \]
\[ r: 2 + 3 = 5 \]

On the other hand, we do not regard sentences such as the exclamation

“What an interesting course!”

or the command

“Get up and do your homework assignments.”

as statements since they do not have truth values (true or false)

The preceding statements represented by \( p, q, \) and \( q \) are considered to be \textbf{primitive}, for there is no way to break them down into anything simpler.
Propositional Connectives

- propositional logic investigates the truth value of compositions of propositions depending on the truth of the components

- only the extensions of the sentences corresponding to propositions are considered

- thus the truth value of a composition only depends on that of the components and on the operations applied

- new statements can be obtained from existing ones in two ways:

  1. **negation:** \( \neg p \)  the negation of \( p \)  
     read “not \( p \)”

  2. combine two statements \( p \) and \( q \) into a **compound** statement, using the following logical connectives

     a) **conjunction:** \( p \land q \)  the conjunction of \( p \) and \( q \)  
        read “\( p \) and \( q \)”

     b) **disjunction:** \( p \lor q \)  the disjunction of \( p \) and \( q \)  
        read “\( p \) or \( q \)”

     c) **implication:** \( p \rightarrow q \)  the implication of \( q \) by \( p \)  
        read “\( p \) implies \( q \)”

     d) **biconditional:** \( p \leftrightarrow q \)  the biconditional of \( p \) and \( q \)  
        read “\( p \) if and only if \( q \)”
Examples:

$p$: COT3100 is a required course

$q$: Students must pass the Foundation Exam

$p \land q$: COT3100 is a required course and students must pass the foundation exam

$p \lor q$: COT3100 is a required course or students must pass the foundation exam

$p \rightarrow q$: If COT3100 is a required course, then students must pass the foundation exam

$p \leftrightarrow q$: COT3100 is a required course if and only if students must pass the foundation exam
Remarks:

- **Disjunction**
  The word “or” is used in the **inclusive** sense. Consequently, $p \lor q$ is true if one or the other of $p$, $q$ is true or if **both** of the statements $p$, $q$ are true.

  In English, we sometime write and/or to point this out.

  The **exclusive** “or” is denoted by $p \lor q$. The compound statement $p \lor q$ is true if one or the other of $p$, $q$ is true but **not both** of the statements $p$, $q$ are true.

- **Implication**
  We say “$p$ implies $q$” and write $p \rightarrow q$ to designate the statement, which is the **implication** of $q$ by $p$. Alternatively, we can also say

  i. “if $p$, then $q$”
  ii. “$p$ is **sufficient** for $q$”
  iii. “$p$ is a **sufficient condition** for $q$”
  iv. “$q$ is a **necessary** for $q$”
  v. “$q$ is a **necessary condition** for $q$”
  vi. “$p$ only if $q$”

  The statement $p$ is called the **hypothesis** of the implication; $q$ is called the **conclusion**
Remark:
Throughout our discussion on logic we must realize that a sentence such as

“The integer $x$ is an integer.”

is not a statement because its truth value (true or false) cannot be determined until a numerical value is assigned for $x$.

If $x$ were assigned the value 7, then the statement would be a true statement. Assigning a value such as $\pi$, however, would make the resulting statement false.

Predicate Calculus deals with such situations.
Truth Tables

In propositional calculus, the propositions $p$ and $q$ are considered as variables (propositional variables) which can have only the values 0 and 1. Then the truth tables contain the truth functions defining the propositional operations.

Negation

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Conjunction/Disjunction/Implication/Biconditional

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Remarks:

- The four possible truth assignments for \( p, q \) can be listed in any order. For later work, the particular order presented here will prove useful.

- We see that the columns of truth values for \( p \) and \( \neg p \) are the opposite of each other.

- The statement \( p \land q \) is true only when both \( p, q \) are true, whereas \( p \lor q \) is false only when both the component statements \( p, q \) are false.

- For the implication \( p \rightarrow q \), the result is true in all cases except where \( p \) is true and \( q \) is false. We do not want a true statement to lead us into believing something that is false. However, we regard as true a statement “if \( 2+3=6 \), then \( 2+4=7 \),” even though the statements “\( 2+3=6 \)” and “\( 2+4=7 \)” are both false.

- The biconditional \( p \leftrightarrow q \) is true when the statements \( p, q \) have the same truth value and is false otherwise.
Formulas in Propositional Calculus

- we can compose compound expressions (formulas) of propositional calculus from the propositional variables in terms of unary (negation) and binary operations (conjunction, disjunction, implication, and biconditional).

- these expressions, that is, formulas, are defined in an inductive way:

  1. propositional variables and the constants 0,1 are formulas

  2. if \( s \) and \( t \) are formulas, then \( (\neg s), (s \land t), (s \lor t), (s \rightarrow t), \) and \( (s \leftrightarrow t) \) are also formulas.

  to simplify formulas we may omit parentheses after introducing precedence rules; in the following sequence every propositional operation bind more strongly than the next one in the sequence:

  \[ \neg \land \lor \rightarrow \leftrightarrow \]

  by these simplifications, for instance the formula \( ((p \lor (\neg q)) \rightarrow ((p \land q) \lor r)) \) can be written more briefly in the form \( p \lor \neg q \rightarrow p \land q \lor r \)

  Often the notation \( p \) is used instead of \( \neg p \) and the symbol \( \land \) is omitted.

  \[ p \lor q \rightarrow pq \lor r \]
Remark
Grimaldi does not use all these simplification rules in his book, so let us not use them to be consistent with the book. The only simplification he uses are

- \( \neg p \) instead of \((\neg p)\)

- \((p \lor (\neg q)) \rightarrow (p \land q) \lor r\) instead of \(((p \lor (\neg q)) \rightarrow (p \land q) \lor r))\), that is, the leftmost and rightmost parentheses can be omitted

Remark
We also refer to expressions (or formulas) as simply statements. We say that \(s\) is a primitive statement if \(s\) is a propositional variable. Otherwise we say that \(s\) is a compound statement.
Truth Functions

- if we assign a truth value to every propositional variable of a formula, we call the assignment an **interpretation** of the propositional variables.

- using definition (truth tables) of propositional operations we can assign a truth value to a formula for every possible interpretation of the variables.

- thus for instance the formula given below determines a truth function of three variables (a **Boolean function**)

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<td>$(p \land \neg q) \rightarrow ((p \land q) \lor r)$</td>
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• In this way, every formula with \( n \) propositional variables determines an \( n \)-place (or \( n \)-ary) function, that is, a function which assigns a truth value to every \( n \)-tuple of truth values

• there are \( 2^{(2^n)} \) \( n \)-ary truth functions, in particular there are \( 16=2^{(2^2)} \) binary ones.
**Examples**

Let $s, t,$ and $u$ denote the following primitive statements:

$s$: Phyllis goes out for a walk.
$t$: The moon is out.
$u$: It is snowing.

The following English statements provide translations for the given (symbolic) compound statements.

a) $(t \land \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk,

b) $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if is not snowing Phyllis goes out for a walk.

c) $\neg (s \leftrightarrow (u \lor t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

**Remark**

Section 2.1 Basic Connectives and Truth Tables contains more examples. Check them out.
Week #1

Lecture # 2

January 11th
Example
The statement $p \rightarrow (p \lor q)$ is true and the statement $p \land (\neg p \land q)$ is false for all truth value assignments (interpretations) for the component statements $p$ and $q$.

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<th>$p \lor q$</th>
<th>$p \rightarrow (p \lor q)$</th>
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This motivates the following definition.

Definition
A compound statement is called a **tautology** if it is true for all truth assignments for its component statements. If a compound statement is false for all such assignments, then it is called a **contradiction**.

We use the symbol $T_0$ to denote any tautology and $F_1$ to denote any contradiction.
Logical Equivalence: The Laws of Logic

- in all areas of mathematics we need to know when the entities we are studying are equal or essentially the same

- for example, in arithmetic and algebra we know that two nonzero real numbers are equal when they have the same magnitude and algebraic sign; hence for two nonzero numbers $x$ and $y$, we have $x = y$ if $|x| = |y|$ and $x \cdot y > 0$

- our study of logic is often referred to as the algebra of propositions (as opposed to the algebra of real numbers); in this algebra we shall use the truth tables of the statements to develop an idea when two such entities are the same

Example
The truth tables for the compound statement $\neg p \lor q$ and $p \rightarrow q$ are exactly the same:

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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg p \lor q$</th>
<th>$p \rightarrow q$</th>
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This situation leads us to the following idea:

**Definition**
Two statements $s_1$ and $s_2$ are said to be **logically equivalent**, and we write $s_1 \iff s_2$, when the statement $s_1$ is true (respectively, false) if and only if the statement $s_2$ is true (respectively, false).

Consequently, we can check the logical equivalence of statements in terms of truth tables.

It is readily checked that

$$p \leftrightarrow q \iff (p \rightarrow q) \land (p \rightarrow q) \iff (\neg p \lor q) \land (\neg q \lor p)$$

Consequently, if we choose so, we can eliminate the connectives $\rightarrow$ and $\iff$ from compound statements.

We also have the equivalence

$$p \lor q \iff (p \lor q) \land \neg (p \land q)$$
Remark

in general, if \( s_1 \) and \( s_2 \) are statements and \( s_1 \leftrightarrow s_2 \) is a tautology, then \( s_1 \) and \( s_2 \) must have the same corresponding truth tables and \( s_1 \leftrightarrow s_2 \).

When \( s_1 \) and \( s_2 \) are logically equivalent, then the compound statement \( s_1 \leftrightarrow s_2 \) is also a tautology. Under these circumstances it is also true that \( \neg s_1 \leftrightarrow \neg s_2 \), and \( \neg s_1 \leftrightarrow \neg s_2 \) is a tautology.

When \( s_1, s_2 \) and \( s_3 \) are statements where \( s_1 \leftrightarrow s_2 \) and \( s_2 \leftrightarrow s_3 \) then \( s_1 \leftrightarrow s_3 \).
Example
We now derive two elementary logical equivalences. To do this, let us consider the table

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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg(p \land q)$</th>
<th>$\neg p \lor \neg q$</th>
<th>$\neg(p \lor q)$</th>
<th>$\neg p \land \neg q$</th>
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Columns 3 and 4 reveal that

$$\neg(p \land q) \iff \neg p \lor \neg q$$

and columns 5 and 6 reveal that

$$\neg(p \lor q) \iff \neg p \land \neg q$$

Using the concepts of logical equivalence, tautology, and contradiction we state the following list of laws for the algebra of propositions (also called **elementary laws of propositional calculus**).
For any statements (primitive or compound) \( p, q, \) and \( r, \) any tautology \( T_0 \) and any contradiction \( F_0: \)

1. Double Negation \[ \neg \neg p \iff p \]

2. De Morgan Laws \[ \neg (p \land q) \iff \neg p \lor \neg q \quad \neg (p \lor q) \iff \neg p \land \neg q \]

3. Commutative Laws \[ p \land q \iff q \land p \quad p \lor q \iff q \lor p \]

4. Associative Laws \[ (p \land q) \land r \iff p \land (q \land r) \quad (p \lor q) \lor r \iff p \lor (q \lor r) \]

5. Distributive Laws \[ (p \lor q) \land r \iff (p \land r) \lor (q \land r) \quad (p \land q) \lor r \iff (p \lor r) \land (q \lor r) \]

6. Idempotent Laws \[ p \land p \iff p \quad p \lor p \iff p \]

7. Identity Laws \[ p \land 1 \iff p \quad p \lor 0 \iff p \]

8. Inverse Laws \[ p \land \neg p \iff 0 \quad p \lor \neg p \iff 1 \]

9. Domination Laws \[ p \land 0 \iff 1 \quad p \lor 1 \iff 1 \]

10. Absorption Laws \[ p \land (p \lor q) \iff p \quad p \lor (p \land q) \iff p \]
Remark
Read Section 2.2. in Grimaldi's book for a proof of these laws and many examples illustrating the intuition behind them.
Logical Implication: Rules of Inference

- we now use the ideas of tautology and implication to describe what we mean by a valid argument

- this will help us develop needed skills for proving mathematical theorems

- in general, an argument starts with a list of given statements called premises and a statement called the conclusion of the argument

- we examine these premises, say $p_1, p_2, p_3, ..., p_n$, and try to show that the conclusion $q$ follows logically from these given statements

- that is, we try to show that if each of $p_1, p_2, p_3, ..., p_n$ is a true statement, then the statement $q$ is also true
• to do so, we examine the implication

\[ p_1 \land p_2 \land p_3 \land \ldots \land p_n \rightarrow q \]

where the hypothesis is the conjunction of the \( n \) premises

• if any one \( p_1, p_2, p_3, \ldots, p_n \) of is false, then no matter what truth value \( q \) has, the implication \( p_1 \land p_2 \land p_3 \land \ldots \land p_n \rightarrow q \) is true

• consequently, if we start with the premises \( p_1, p_2, p_3, \ldots, p_n \) – each with truth value 1 – and find that under these circumstances \( q \) also has the value 1, then the implication

\[ p_1 \land p_2 \land p_3 \land \ldots \land p_n \rightarrow q \]

is a **tautology** and we have a valid argument
Example
Let \( p, q, r \) denote the primitive statements given as

\[
\begin{align*}
  p: & \quad \text{Roger studies} \\
  q: & \quad \text{Roger parties too much} \\
  r: & \quad \text{Roger passes the foundation exam}
\end{align*}
\]

Now let \( p_1, p_2, p_3 \), denote the premises

\[
\begin{align*}
  p_1: & \quad \text{If Roger studies, then he will pass the foundation exam} \quad p \rightarrow r \\
  p_2: & \quad \text{If Roger doesn't party too much, then he'll study} \quad \neg q \rightarrow p \\
  p_3: & \quad \text{Roger failed the foundation exam} \quad \neg r
\end{align*}
\]

We want to determine whether the argument

\[
(p_1 \land p_2 \land p_3) \rightarrow q
\]

is a valid argument. To do so, we have to examine the truth table for the implication

\[
(((p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r) \rightarrow q
\]
The truth table for the implication \(((p \to r) \land (\neg q \to p) \land \neg r) \to q\) is:

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<th>(\neg q \to p)</th>
<th>(\neg r)</th>
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<td>1</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Because the last column contains all 1's, the implication is a tautology. Hence we can say that \(((p \to r) \land (\neg q \to p) \land \neg r) \to q\) is a valid argument.
The ideas presented in the preceding example lead to the following

**Definition**
If \( p, q \) are arbitrary statements such that \( p \rightarrow q \) is a tautology, then we say that \( p \) **logically implies** \( q \) and we write \( p \implies q \) to denote the situation.

When \( p, q \) are statements and \( p \implies q \), then the implication \( p \rightarrow q \) is a tautology and we refer to \( p \rightarrow q \) as a **logical implication**.

Note that we can avoid dealing with the idea of a tautology here by saying that \( p \implies q \) (that is, \( p \) logically implies \( q \)) if \( q \) is true whenever \( p \) is true.

It would be very inconvenient if we had to construct “big” truth tables each time we wanted to check whether an implication is a valid argument or not. Fortunately, there are **rules of inference** which make this task easier.
As a first example we consider the rule of inference called **Modus Ponens**, or the **Rule of Detachment**. In symbolic form this rule is expressed by the logical implication

\[(p \land (p \rightarrow q)) \rightarrow q\]

which is verified by constructing the corresponding truth table

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>q</td>
<td>p→q</td>
<td>p ∧ (p→q)</td>
<td>(p ∧ (p→q)) → q</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>-----</td>
<td>------------</td>
<td>-----------------</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

The actual rule will be written in the tabular form

\[
p
\begin{array}{c}
p→q \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\hline
\downarrow
\end{array}
\]

\[
q
\]

where the symbol ▲ stands for the word “therefore,” indicating that q is the conclusion for the premises p and p→q, which appear above the horizontal line.
This rule arises when we argue that if (1) $p$ is true, and (2) $p \rightarrow q$ is true (or $p \Rightarrow q$), then the conclusion $q$ must also be true.

The following valid argument shows how to apply the Rule of Detachment

1. My students study the class material.
2. If my students study the class material, then they will get good grades.

\[ \text{Therefore, my students will get good grades.} \]

The table on the next page contains some of the rules of inference. Table 2.19 on page 78 in Grimaldi's book for all rules of inference. Section 2.3 Logical Implication: Rules of Inference contains the proofs of these rules and many examples illustrating the intuition underlying these rules.
<table>
<thead>
<tr>
<th>Rule of Inference</th>
<th>Related Logical Implication</th>
<th>Name of Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( (p \land (p \rightarrow q)) \rightarrow q )</td>
<td>Rule of Detachment (Modus Ponens)</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td>( p )</td>
<td></td>
</tr>
<tr>
<td>( q )</td>
<td>( ((p \rightarrow q) \land (q \rightarrow r)) \rightarrow r )</td>
<td>Law of the Syllogism</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td>( q \rightarrow r )</td>
<td></td>
</tr>
<tr>
<td>( q \rightarrow r )</td>
<td>( p \rightarrow r )</td>
<td></td>
</tr>
<tr>
<td>( \neg q )</td>
<td>( \neg p )</td>
<td></td>
</tr>
</tbody>
</table>
Tautologies, Inferences in Mathematics

- recall that a formula in propositional logic is said to be a **tautology** if the value of its truth function is identical to 1

- consequently, two formulas $p$ and $q$ are called **logically equivalent** if the formula $p \leftrightarrow q$ is a tautology

- laws of propositional calculus (elementary laws and inference laws) often reflect inference methods used in mathematics

- as an example, consider the **law of contraposition**, that is, the tautology

  $$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$$

- this law, which also has the form

  $$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$$

  can be interpreted in the following way: to show that $q$ is a consequence of $p$ is the same as showing that $\neg q$ is a consequence of $\neg p$
Indirect proof means the following principle: to show that $q$ is a consequence of $p$, we suppose $q$ to be false, and under the assumption that $p$ is true, we derive a contradiction.

This principle can be formalized in propositional calculus in several ways:

\[
(p \rightarrow q) \iff (p \land \neg q) \rightarrow \neg p
\]

\[
(p \rightarrow q) \iff (p \land \neg q) \rightarrow \neg q
\]

\[
(p \rightarrow q) \iff (p \land \neg q) \rightarrow 0
\]

We will learn more about proof methods in mathematics later.
Predicate Calculus

- for developing the logical foundations of mathematics we need a logic which has a **stronger expressive power** than propositional calculus
- to describe properties of most of the objects in mathematics and the relations between these objects the **predicate calculus** is needed

Motivation

- sentences that involve variables are not statements; for example, the sentence “\(x+2\) is an even integer” is not necessarily true or false unless we know what value is substituted for the variable \(x\)
- if we restrict our attention to integers, then when \(x\) is substituted by -5, -1, or 3, for instance, the resulting statement is false
- in fact, it is false (true) whenever \(x\) is replaced by an odd (even) integer
Predicates

- we include the objects to be investigated into a set $X$ called the **universe** (or **universe of discourse**); for example, this domain could be the set $\mathbb{N}$ of natural numbers

- the properties of the objects such as “$n$ is a prime” or “$n$ is even”, and the relations between objects such as “$m$ is smaller than $n$” or “$m$ is greater than $n$” are called **predicates**

- an $n$-place **predicate** $p$ over the domain $X$ is an assignment

$$p : X^n \to \{0,1\}$$

which assigns a truth value to every $n$-tuple of the objects

- so the predicates introduced above on natural numbers are a one-place (or unary) predicate and a two-place (or binary) predicates
Quantifiers

- a characteristic feature of predicate logic is the use of **quantifiers**:
  - **universal quantifier** or “for every” quantifier $\forall$
  - **existential qualifier** or “for some” qualifier $\exists$

- if $p$ is a unary predicate, then
  - the sentence “$p(x)$ is true for every $x$ in $X$” is denoted by $\forall x \ p(x)$
  - the sentence “$p(x)$ is true for some $x$ in $X$” is denoted by $\exists x \ p(x)$

- applying a quantifier to the unary predicate $p$, we get a sentence

- for example, if the universe is the set of natural numbers and $p$ denotes the (unary) predicate “$n$ is prime”, then $\forall n \ p(n)$ is false and $\exists n \ p(n)$ is a true

- $\forall n \ p(n)$ is false because 10 is not prime; thus not all natural numbers are prime

- $\exists n \ p(n)$ is true because 5 is prime; thus there is at least one natural number that is prime
Formulas in Predicate Logic

The formulas in predicate logic are defined in an inductive way:

1. If $x_1, \ldots, x_n$ are variables (variables taking on values from the universe) and $p$ is an $n$-place predicate symbol, then $p(x_1, \ldots, x_n)$ is a formula (elementary formula).

2. If $A$ and $B$ are formulas, then $\neg A, (A \land B), (A \lor B), (A \rightarrow B), (A \leftrightarrow B), (\forall x \ A), (\exists x \ A)$ are also formulas.

- An occurrence of a variable $x$ is said to be bound in a formula if $x$ is a variable in $\forall x$ or $\exists x$ or the occurrence of $x$ is in the scope of these types of quantifiers; otherwise an occurrence of $x$ is free in this formula.

- A formula of predicate logic which does not contain any free occurrences of individual variables is said to be a closed formula.
Example
Consider the formula $\forall x \ q(x, y)$

- the first occurrence of $x$ is bound as it occurs in $\forall x$
- the second occurrence of $x$ is bound as it is in the scope of $\forall x$
- the occurrence of $y$ is not bound

Example
Consider the formula $(\forall x \exists y \ q(x, y)) \rightarrow p(x)$

- the third occurrence of $x$ is not bound; it is not in the scope of $\forall x$
- the second occurrence of $y$ is bound as it is in the scope of $\exists y$

Remark
Considering a propositional variable to be a null-place predicate, we can consider propositional calculus as a part of predicate calculus.
Example:

Let the universe $X$ be the set $\mathbb{R}$ of real numbers. Consider the open (or free) statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ given by

$$p(x) : x > 0$$
$$q(x) : x^2 > 0$$
$$r(x) : x^2 - 3x - 4 = 0$$

The following statements are true

1. $\exists x : (p(x) \land r(x))$
   It is true because the real number 4 is a member of the universe and is such that $p(4)$ and $r(x)$ are true as we have $4 > 0$ and $4^2 - 3 \cdot 4 - 4 = 16 - 12 - 4 = 0$.

2. $\forall x : (p(x) \rightarrow q(x))$

The following statement is false $\forall x : (q(x) \rightarrow p(x))$ because $q(-\pi)$ is true but $p(-\pi)$ is false.
Interpretation of Predicate Calculus Formulas

An interpretation of predicate calculus is a pair
- a set (the universe) and
- an assignment, which assigns an $n$-place predicate to every $n$-ary predicate symbol.

For every prefixed value of free variables the concept of the truth evaluation of a formula is similar to the propositional case. The truth value of a closed formula is 0 or 1.

With a formula with free variables, we can associate the values of objects for which the truth evaluation of the formula is true; the values constitute a relation on the universe.
**Example**
Let \( p \) denote the two-place relation \( \leq \) on the set \( \mathbb{N} = \{0,1,2,3,\ldots\} \) of natural numbers.
Then

- \( p(x, y) \) characterizes the set of all pairs \( (x, y) \) with \( x \leq y \) (two place or binary relation on \( \mathbb{N} \)); here \( x \) and \( y \) are free variables

- \( \forall y \ p(x, y) \) characterizes the subset of \( \mathbb{N} \) (unary relation) consisting of the element 0 only; here \( x \) is a free variable and \( y \) is a bound variable

- \( \exists x \ \forall y \ p(x, y) \) corresponds to the sentence “There is a smallest natural number”; the truth value is true; here \( x \) and \( y \) are bound

**Example**
Let \( q \) denote the two-place relation \( \leq \) on the set \( \mathbb{Z} = \{0,1,-1,2,-1,3,-3,\ldots\} \) of integers.
Why is the statement \( \exists x \ \forall y \ q(x, y) \) false?

There is no smallest integer!!! However, the smallest natural number is 0.

What about the statement \( \forall y \exists x \ q(x, y) \) false?
<table>
<thead>
<tr>
<th>Statement</th>
<th>When Is it True?</th>
<th>When Is it False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists x \ p(x) )</td>
<td>for <strong>some</strong> (at least one) ( a ) in the universe, ( p(a) ) is true</td>
<td>for <strong>every</strong> ( a ) in the universe, ( p(a) ) is false</td>
</tr>
<tr>
<td>( \forall x \ p(x) )</td>
<td>for <strong>every</strong> replacement ( a ) in the universe, ( p(a) ) is true</td>
<td>there is <strong>at least one</strong> replacement ( a ) in the universe for which ( p(a) ) is false</td>
</tr>
<tr>
<td>( \exists x \ \neg p(x) )</td>
<td>for a least one choice ( a ) in the universe, ( p(a) ) is false, so its negation ( \neg p(a) ) is true</td>
<td>for <strong>every</strong> ( a ) in the universe, ( p(a) ) is true</td>
</tr>
<tr>
<td>( \forall x \ \neg p(x) )</td>
<td>for <strong>every</strong> replacement ( a ) in the universe, ( p(a) ) is false</td>
<td>there is <strong>at least one</strong> replacement ( a ) in the universe for which ( p(a) ) is true</td>
</tr>
</tbody>
</table>
Logical Equivalences

Definition
Let \( p(x) \), \( q(x) \) be open statements (that is, the occurrence of \( x \) is free) defined for a given universe.

The open statements \( p(x) \) and \( q(x) \) are called (logically) equivalent, and we write

\[
\forall x \ (p(x) \iff q(x))
\]

when the biconditional \( p(a) \iff q(a) \) is true for each replacement \( a \) in the universe (that is, \( p(a) \iff q(a) \) for each \( a \) in the universe).

If the implication \( p(a) \rightarrow q(a) \) is true for each \( a \) in the universe (that is, \( p(a) \Rightarrow q(a) \) for each \( a \) in the universe), then we write

\[
\forall x \ (p(x) \Rightarrow q(x))
\]

and say that \( p(x) \) logically implies \( q(x) \).

Read Section 2.4!!!
Logically Valid Formulas

Definition
A formula is said to be **logically valid** (or a **tautology**) if it is true for every interpretation.

The negation of formulas is characterized by the identities below

\[-\forall x \ p(x) \iff \exists x \ \neg p(x) \quad \text{or} \quad \neg \exists x \ p(x) \iff \forall x \ \neg p(x)\]

Using these identities the quantifiers \( \forall \) and \( \exists \) can be expressed in terms of each other

\[-\forall x \ p(x) \iff \neq \exists x \ \neg p(x) \quad \text{or} \quad \exists x \ p(x) \iff \neg \forall x \ \neg p(x)\]

Further equivalences of the predicate calculus are

\[\forall x \ \forall y \ p(x, y) \iff \forall y \ \forall x \ p(x, y)\]

\[\exists x \ \exists y \ p(x, y) \iff \exists y \ \exists x \ p(x, y)\]

\[\forall x \ p(x) \land \ \forall x \ q(x) \iff \forall x \ (p(x) \land q(x))\]

\[\exists x \ p(x) \lor \ \exists x \ q(x) \iff \exists x \ (p(x) \lor q(x))\]
The following implications are also valid:

\[ \forall x \, p(x) \lor \forall x \, q(x) \Rightarrow \forall x \, (p(x) \lor q(x)) \]

\[ \exists x \, (p(x) \land q(x)) \Rightarrow \exists x \, p(x) \land \exists x \, q(x) \]

\[ \forall x \, (p(x) \rightarrow q(x)) \Rightarrow (\forall x \, p(x) \rightarrow \forall x \, q(x)) \]

\[ \forall x \, (p(x) \leftrightarrow q(x)) \Rightarrow (\forall x \, p(x) \leftrightarrow \forall x \, q(x)) \]

\[ \exists x \, \forall y \, p(x, y) \Rightarrow \forall y \, \exists x \, p(x, y) \]

The converses of these implications are not valid, in particular, we have to be careful with the fact that the quantifiers \( \forall \) and \( \exists \) do not commute (the converse of the last implication is false).
The Rule of Universal Specification

If an open statement becomes true for all replacements by the members in a given universe, then that open statement is true for each specific individual member in that universe.

A bit more symbolically: if $p(x)$ is an open statement for a given universe, and if $\forall x \ p(x)$ is true, then $p(a)$ is true for each $a$ in the universe.

Example
For the universe of all people, consider the open statements

$$m(x): \text{x is a mathematics professor} \quad c(x): \text{x has studied calculus}$$

Now consider the following argument:

All mathematics professor have studied calculus.

Leona is a mathematics professor.

Therefore Leona has studied calculus.
If we let $L$ represent this particular woman (in our universe) named Leona, then we can rewrite this argument in symbolic form

$$\forall x (m(x) \rightarrow c(x))$$
$$m(L)$$

\[\text{-----------------------------}\]
\[\text{▲} \quad c(L)\]

Pages 106 – 110 contain more examples. Chem them out!!!
The Rule of Universal Generalization

If an open statement $p(x)$ is proved to be true when $x$ is replaced by any arbitrarily chosen element $c$ from our universe, then the universally quantified statement $\forall x \ p(x)$ is true.

Furthermore, this rule extends beyond a single variable.

So if, for example, we have an open statement $q(x,y)$ that is proved to be true when $x$ and $y$ are replaced by arbitrarily chosen elements of the same universe, or their own respective universes, then the universally quantified statement $\forall x \ \forall y \ q(x,y)$ is true.

Similar results hold for three or more variables.
Example
Let \( p(x), q(x), r(x) \) be open statements that are defined for a given universe. We show that the argument

\[
\forall x(p(x) \rightarrow q(x)) \\
\forall x(q(x) \rightarrow r(x))
\]

\[\Delta \quad \forall x(p(x) \rightarrow r(x))\]

is valid by considering the following.

<table>
<thead>
<tr>
<th>Steps</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \forall x(p(x) \rightarrow q(x)) )</td>
<td>Premise</td>
</tr>
<tr>
<td>2. ( p(c) \rightarrow q(c) )</td>
<td>Step (1) and the Rule of Universal Specification</td>
</tr>
<tr>
<td>3. ( \forall x(q(x) \rightarrow r(x)) )</td>
<td>Premise</td>
</tr>
<tr>
<td>4. ( q(c) \rightarrow r(c) )</td>
<td>Step (3) and the Rule of Universal Specification</td>
</tr>
<tr>
<td>5. ( p(c) \rightarrow r(c) )</td>
<td>Steps (2) and (4) and the Law of Syllogism</td>
</tr>
<tr>
<td>6. ( \Delta \quad \forall x(p(x) \rightarrow q(x)) )</td>
<td>Step (5) and the Rule of Universal Generalization</td>
</tr>
</tbody>
</table>

Here the element \( c \) introduced in steps (2) and (4) is the same specific but arbitrarily chosen element from the universe. Since this element \( c \) has no special or distinguishing properties but does share all the common features of every other element in the universe, we can use the Rule of Universal Generalization to go from step (5) to step (6).
Example

Let Paul - represented by $P$ - be a particular man (in our universe) who has not studied calculus. Is his a mathematics professor? No! But why?

Recall that $s \rightarrow t \Leftrightarrow \neg t \rightarrow \neg s$ for arbitrary (propositional) statements $s$ and $t$. Therefore

$$\forall x (m(x) \rightarrow c(x)) \Leftrightarrow \forall x (\neg c(x) \rightarrow \neg m(x))$$

This is proved by first applying the Rule of Universal Specification and then the Rule of Universal Generalization (just as we did in the previous example).

Now using the rule of universal specification

$$\forall x (\neg c(x) \rightarrow \neg m(x))$$

$$\neg c(P)$$

$$\neg m(P)$$

we see that Paul is not a mathematics professor.

More generally, “the (propositional) rules of inference generalize to predicate rules of inference.”
Restricted Quantification

Often it is useful to restrict quantification to a subset of a given universe. So we consider

\[ \forall x \in X \ p(x) \ \text{as a shorthand notation of} \ \forall x \ (x \in X \to p(x)) \]

\[ \exists x \in X \ p(x) \ \text{as a shorthand notation of} \ \exists x \ (x \in X \to p(x)) \]
Set Theory

Concept of Set, Special Set

The founder of set theory is Georg Cantor (1845-1918). The importance of the notion introduced by him became well known only later. Set theory has a decisive role in all branches of mathematics, and today it is an essential tool of mathematics and its applications.
Membership relation

Sets and their Elements

The fundamental notion of set theory is the membership relation. A set $A$ is a collection of certain things $a$ (objects, ideas, etc.) that we think belong together for certain reasons. These objects are called the elements of the set.

We write

\[ a \in A \quad \text{or} \quad a \notin A \]

to denote “$a$ is an element of $A$” and “$a$ is not an element of $A$”, respectively.

Sets can be given by enumerating their elements in braces; for example

\[ M = \{ a, b, c \} \quad \text{or} \quad U = \{1, 3, 5, \ldots\} , \]

or by a defining property possessed exactly by the elements of the set.

For instance the set $U$ of odd natural numbers is defined and denoted by

\[ U = \{ x \mid x \text{ is an odd number} \} \]
For number domains the following notation is generally used:

\[ \mathbb{N} = \{0, 1, 2, \ldots\} \quad \text{set of natural numbers} \]

\[ \mathbb{Z} = \{0, 1, -1, 2, -2 \ldots\} \quad \text{set of the integers} \]

\[ \mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z} \land q \neq 0\} \quad \text{set of the rational numbers} \]

\[ \mathbb{R} \quad \text{set of the real number} \]

\[ \mathbb{C} \quad \text{set of the complex number} \]

**Principle of Extensionality for Sets**

Two sets \( A \) and \( B \) are identical if and only if they have the same elements, that is,

\[ A = B \iff \forall x (x \in A \iff x \in B) \]

The set \{3,1,3,7,2\} and \{1,2,3,7\} are the same. A set contains every element only “once”, even if it enumerate several times.
Subsets

Definition
If $A$ and $B$ are sets and

$$\forall x \left( x \in A \Rightarrow x \in B \right)$$

holds, then $A$ is called a subset of $B$, and this is denoted by $A \subseteq B$. In words $A$ is a subset of $B$, if all elements of $A$ also belong to $B$.

If for $A \subseteq B$ there are some elements in $B$ such that they are not in $A$, then we call $A$ a proper subset of $B$, and denote it by $A \subset B$.

Obviously, every set is a subset of itself $A \subseteq A$.

Example
Suppose $A=\{2,4,6,8,10\}$ is a set of even numbers and $B=\{1,2,3,4,5,6,7,8,9,10\}$. Since $A$ does not contain odd numbers, $A$ is a proper subset of $B$. 
Definition
It is important and useful to introduce the notion of **empty set** or **void set**, \( \emptyset \), which has no element. Because of the principle of extensionality, there exists only one empty set.

Examples
The set \( \{ x \mid x \in \mathbb{R} \land x^2 + 2x + 2 = 0 \} \) is empty.

\( \emptyset \subseteq M \) for every set \( M \), that is, the empty subset \( \emptyset \) is subset of set \( M \).

For a set \( A \) the empty set and \( A \) itself are called the **trivial subsets** of \( A \).

Definition
Equality of sets: Two sets are equal if and only if both are subsets of each other:

\[ A = B \iff A \subseteq B \land B \subseteq A \]

This fact is often used to prove that two sets are identical.
Definition
The set of all subsets $A$ of $M$ is called the power set of $M$ and is denoted by $P(M)$, that is
$P(M)=\{A \mid A \subseteq M\}$.

Example
For the set $M=\{a,b,c\}$ the power set is
$P(M)=\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$.

It is true that
1. If a set $M$ has $m$ elements, its power set $P(M)$ has $2^m$ elements.
2. For every set $M$ we have $\emptyset, M \in P(M)$, that is $M$ itself and the empty set are elements of the power set.

Definition
The number of elements of a finite set $M$ is called the cardinal number (or cardinality) of $M$ and it is denoted by $|M|$.

Note that the cardinal number of sets with infinitely elements can also be defined.
Operations with Sets

Definition
The graphical representation of sets and set operations are the so-called Venn-diagrams, when we represent sets by plane figures.

So, in the figure on the left, we represent the subset relation \( A \subseteq B \).
**Definition**

By set operations we form new sets from the given sets in different ways by

- **Union:** Let \( A \) and \( B \) be two sets. The **union set** or **union**, denoted by \( A \cup B \) is defined by

\[
A \cup B = \{ x \mid x \in A \lor x \in B \} 
\]

We say “A union B” or “A cup B”. If \( A \) and \( B \) are given by properties \( p \) and \( q \) respectively, the union set \( A \cup B \) has the elements possessing at least one of the properties.

![Venn diagram showing \( A \cup B \)]

**Example** \( \{1,2,3\} \cup \{2,3,5,6\} = \{1,2,3,5,6\} \)
**Intersection:** Let $A$ and $B$ be two sets. The *intersection set, intersection, cut or cut set*, denoted by $A \cap B$ is defined by

$$A \cap B = \{ x \mid x \in A \land x \in B \}$$

We say “$A$ intersected by $B$” or “$A$ cap $B$”. If $A$ and $B$ are given by properties $p$ and $q$ respectively; the intersection $A \cap B$ has elements possessing both properties $p$ and $q$.

**Example**

$$\{1,2,3\} \cap \{2,3,5,6\} = \{2,3\}$$
Definition
Two sets $A$ and $B$ are called \textbf{disjoint} if they have no common element; for them

$$A \cap B = \emptyset$$

holds, that is their intersection is the empty set.

Example
The set of odd numbers and the set of even numbers are disjoint; their intersection is the empty set, that is

$$\{\text{odd numbers}\} \cap \{\text{even number}\} = \emptyset$$
- **Complement:** If we consider only the subsets of a given set $M$, then the complementary set or the complement $C_M(A)$ of $A$ with respect to $M$ contains all elements of $M$ not belonging to $A$:

$$C_M(A) = \{ x | x \in M \land x \notin A \}$$

We say “complement of $A$ with respect to $M$”, and $M$ is called the **fundamental set** or the **universal set**. If the fundamental set $M$ is obvious from the considered problem, the notation $\bar{A}$ is also used for the complementary set.
Fundamental Laws of Set Algebra

These set operations have analogous properties to the operations in propositional logic. The fundamental laws of set algebra are:

1. Double Complement \( A = A \)
2. De Morgan Laws \( A \cap B = A \cup B \) \( A \cup B = A \cap B \)
3. Commutative Laws \( A \cap B = B \cap A \) \( A \cup B = B \cup A \)
4. Associative Laws \( (A \cap B) \cap C = A \cap (B \cap C) \) \( (A \cup B) \cup C = A \cup (B \cup C) \)
5. Distributive Laws \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)
   \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)
6. Idempotent Laws \[ A \cap A = A \quad A \cup B = B \cup A \]

7. Identity Laws \[ A \cap U = A \quad A \cup \emptyset = A \]

8. Inverse Laws \[ A \cap \overline{A} = \emptyset \quad A \cup \overline{A} = U \]

9. Domination Laws \[ A \cap \emptyset = \emptyset \quad A \cup U = U \]

10. Absorption Laws \[ A \cap (A \cup B) = A \quad A \cup (A \cap B) = A \]

Remark
This table can also be obtained from the fundamental laws of propositional calculus; if we make the following substitutions:

\[
\begin{array}{ll}
\wedge & \text{by} \quad \cap \\
\vee & \quad \cup \\
\neg & \\
0 & \emptyset \\
1 & U
\end{array}
\]

This is not a coincidence; it can be shown that the algebra of propositional logic is isomorphic to the set algebra (we won't pursue this any further).