

## Set Theory (repetition)

Set theory starts with some undefined but intuitively clear terms, then formal logic is used to prove properties about sets and their relationships. Set theory has been used as the foundation for all mathematical theories.

Intuitively, a *set* means *a collection of things (objects)*. The relationship between the objects contained in the set, called the *elements*, and the set, is the *belongs to* relationship. Formally,

let  $A$  denote a set, an element  $x$  belongs to  $A$  is denoted  $x \in A$ ; otherwise,  $x \notin A$  if  $x$

doesn't belong to  $A$ .

There are two ways to describe a set, by an enumeration of its elements or by stating the properties that the elements must satisfy:

**Example:** The set of all even, non-negative integers can be described as

$$E = \{0, 2, 4, \dots\}; \text{ or}$$

$$E = \{x \mid x \geq 0 \text{ and } x = 2n, \text{ where } n \text{ is an integer}\}.$$

After the undefined terms are understood, we can make formal definitions and prove theorems.

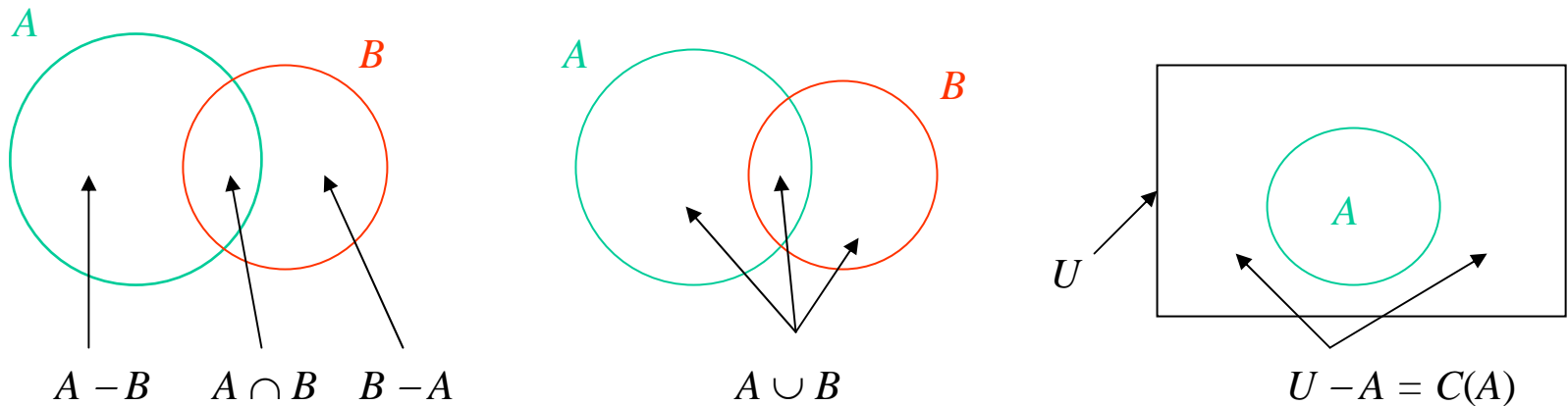
**Definition.** Let  $A, B$  denote two sets.

(a)  $A$  is a subset of  $B$ , denoted  $A \subseteq B$ , if for all  $x \in A$ ,  $x \in B$  is true.  $A \subseteq B$  can also be written as  $B \supseteq A$ .

(b)  $A = B$  if both  $A \subseteq B$  and  $B \subseteq A$ ; that is, for all  $x$ ,  $x \in A \Leftrightarrow x \in B$ , where the symbol  $\Leftrightarrow$  means implication in both directions ( $\Rightarrow$  and  $\Leftarrow$ ), and is called *if and only if* (abbreviated as *iff*).

- (c) The *union* of sets  $A$  and  $B$ , denoted  $A \cup B$ , is defined as  $\{x \mid x \in A \text{ or } x \in B\}$ .
- (d) The *intersection* of sets  $A$  and  $B$ , denoted  $A \cap B$ , is defined as  $\{x \mid x \in A \text{ and } x \in B\}$ .
- (e) The *difference* of sets  $A$  and  $B$ , denoted  $A - B$ , is defined as  $\{x \mid x \in A \text{ and } x \notin B\}$ . (Note that in general,  $A - B \neq B - A$ .)
- (f) The set containing no elements is the empty set, denoted  $\emptyset$ . Typically, the sets under consideration are subsets of a universe  $U$ . The difference between  $U$  and a set  $A$ ,  $U - A$ , is called the complement of  $A$  (denoted  $\neg A$ ).

Sets and their relationships are typically depicted by the *Venn diagrams*:



Many properties of the sets can be proved by the definitions and the use of propositional logic.

**Theorem.** Let  $A, B, C$  denote arbitrary sets. The following properties hold.

(a) (Commutative Law)  $A \cup B = B \cup A, A \cap B = B \cap A$ .

(b) (Associative Law)  $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$ .

(c) (Distributive Law)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

(d) (Idempotent Property)  $A \cup A = A, A \cap A = A$ .

(e) (De Morgan's Law)  $\neg(A \cup B) = \neg A \cap \neg B, \neg(A \cap B) = \neg A \cup \neg B$ .

(f) (Double Negation)  $\neg(\neg A) = A$ .

(g) (Complementary Property)  $A \cap \neg A = \emptyset, A \cup \neg A = U$ , where  $U$  denotes the universe.

**Proof:** We will only prove a few of these properties to demonstrate the ideas.

To prove (a), notice that  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ , by definition

$= \{x \mid x \in B \text{ or } x \in A\}$ , by the commutative law for logical *or*

$= B \cup A$ , by definition.

Similarly,  $A \cap B = \{x \mid x \in A \text{ and } x \in B\} = \{x \mid x \in B \text{ and } x \in A\} = B \cap A$ , using the definition of set intersection and the commutative property of logical *and*.

To prove (d), notice that  $A \cup A = \{x \mid x \in A \text{ or } x \in A\}$ , by definition of  $\cup$

$= \{x \mid x \in A\} = A$ , because for any proposition  $p, p \text{ or } p \equiv p$ .

Let us prove a few more theorems about sets. The main techniques used are the definitions, other theorems, and logical reasoning.

**Theorem.** Let  $A$  and  $B$  denote arbitrary sets. The following properties hold.

- (a)  $A \cap B \subseteq A$ . (**Proof:** By the definition of  $\subseteq$ , we need to prove that if  $x \in A \cap B$ , then  $x \in A$ . By the definition of  $A \cap B$ ,  $x \in A \cap B$  implies  $x \in A$  and  $x \in B$ . In particular,  $x \in A$  is true.)
- (b)  $A \subseteq A \cup B$ . (**Proof:** By the definition of  $\subseteq$ , we need to prove that if  $x \in A$ , then  $x \in A \cup B$ ; that is, prove  $x \in A$  or  $x \in B$ . But this is true by the definition of *or*, since  $x \in A$  is true.)
- (c)  $A \cap \emptyset = \emptyset$ . (**Proof:** By part (a),  $A \cap \emptyset \subseteq \emptyset$ . On the other hand, we proved on page 1-1 that  $\emptyset$  is a subset of any set; in particular,  $\emptyset \subseteq A \cap \emptyset$  is true. Thus,  $A \cap \emptyset = \emptyset$  by the definition of set equality  $=$ .)
- (d)  $A \cup \emptyset = A$ . (**Proof:** By part (b),  $A \subseteq A \cup \emptyset$ . Thus, to prove the two sets  $A$  and  $A \cup \emptyset$  are equal, it suffices to prove that  $A \cup \emptyset \subseteq A$ , that is, to prove that if  $x \in A \cup \emptyset$ , then  $x \in A$ . When  $x \in A \cup \emptyset$ , we have  $x \in A$  or  $x \in \emptyset$ . Since  $x \in \emptyset$  is always false,  $x \in A$  must be true.)
- (e)  $A \subseteq B$  iff  $\neg B \subseteq \neg A$ . (**Proof:**  $A \subseteq B$  iff for all  $x$  ( $x \in A \Rightarrow x \in B$ ), by definition of  $\subseteq$ ;  
iff for all  $x$  ( $x \notin B \Rightarrow x \notin A$ ), by the contrapositive law; iff  $\neg B \subseteq \neg A$ , because  $x \notin B$  means  $x \in \neg B$  and  $x \notin A$  means  $x \in \neg A$ , and by the definition of  $\subseteq$ .)
- (f)  $A \subseteq B$  iff  $A \cap \neg B = \emptyset$ . (**Proof:** By definition,  $A \subseteq B$  iff for all  $x$  ( $x \in A \Rightarrow x \in B$ ) is true. By the contrapositive law (page 1-13), for all  $x$  ( $x \in A \Rightarrow x \in B$ ) iff for all  $x$  ( $x \notin A$  or  $x \in B$ ), which is equivalent to for all  $x$   $\neg(x \in A$  and  $x \notin B$ ) by De Morgan's law; and it is equivalent to  $\neg$ (there exists  $x$  ( $x \in A$  and  $x \notin B$ )), by the De Morgan's law. This last statement is equivalent to saying there is no  $x$  which is in set  $A$  and in set  $\neg B$ ; thus,  $A \cap \neg B = \emptyset$  is true.)

Let us introduce a few more definitions in set theory.

**Definition.** Let  $A$  be a set. The *power set* of  $A$ , denoted  $\text{Power}(A)$  or  $2^A$ , is the set of all subsets of  $A$ .

**Example.** Let  $A = \{1, 2\}$ . Then  $\text{Power}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ , a set of the 4 subsets of  $A$ .

**Definition.** Let  $A$  and  $B$  denote two sets. Define the *Cartesian product* of  $A$  and  $B$  as  $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ , where the notation  $(a, b)$  denotes an ordered pair (or a 2-tuple). Note that the ordering of the elements in an ordered pair is significant, that is, two ordered pairs are equal,  $(a, b) = (c, d)$  iff  $a = c$  and  $b = d$ . More generally, the Cartesian product of  $n$  sets  $A_1, A_2, \dots, A_n$ , is defined as  $\prod_{i=1}^n A_i = \{(a_1, \dots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$  where  $(a_1, \dots, a_n)$  defines an (ordered)  $n$ -tuple.

**Example.** Let  $A = \{1, 2\}$  and  $B = \{a, b\}$ . Then  $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$ .

One application of the set theory concerns the counting problems. There are two basic counting principles, dealing with set union and set product.

**(The Sum Principle)** Let  $A$  and  $B$  denote two finite sets and  $A \cap B = \emptyset$ . Then  $|A \cup B| = |A| + |B|$ . More generally, let  $A_1, A_2, \dots, A_n$  denote  $n$  finite sets,  $n \geq 1$ , and these sets are mutually disjoint, that is,  $A_i \cap A_j = \emptyset$  for  $i \neq j$ . Then  $|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n| = \sum_{i=1}^n |A_i|$

**(The Product Principle)** Let  $A_1, A_2, \dots, A_n$  denote  $n$  finite sets. Then  $|\prod_{i=1}^n A_i| = \prod_{i=1}^n |A_i|$

**Example.** Let  $A = \{1, 2, 3\}$  and  $B = \{4, 5\}$  denote two sets. Then  $|A \cup B| = 5 = |A| + |B|$ , because  $A \cap B = \emptyset$ ;  $|A \times B| = |\{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}| = 6 = 2 \cdot 3 = |A| |B|$ .

When the sets  $A$  and  $B$  are not disjoint, the following result tells how to count  $|A \cup B|$ .

**Theorem.** Let  $A$  and  $B$  denote two finite sets. Then  $|A \cup B| = |A| + |B| - |A \cap B|$ .

**Proof:** Since each element of  $A \cup B$  belongs to either  $A$  or  $B$ , the sum  $|A| + |B|$  includes a count for each of the elements of  $A \cup B$ , but those elements of  $A \cap B$  are counted twice. Thus,  $|A| + |B| - |A \cap B|$  counts each element of  $A \cup B$  exactly once, that is, it is equal to  $|A \cup B|$ . More precisely, we first claim that the following is a disjoint union:

$$A = (A - B) \cup (A \cap B) \tag{1}$$

Thus, by the definition of set equality, we want to prove that

$$A \subseteq (A - B) \cup (A \cap B) \tag{2}$$

$$(A - B) \cup (A \cap B) \subseteq A \tag{3}$$

and  $(A - B) \cap (A \cap B) = \emptyset \tag{4}$

To prove (2), let  $x \in A$ . Since either  $x \in B$  or  $x \notin B$  is true: in the former case,  $x \in A \cap B$  by definition, and in the latter case, we have  $x \in A$  and  $x \notin B$ , which means  $x \in A - B$  by definition.

To prove (3), note that  $A - B \subseteq A$  because each  $x \in A - B$  must also have  $x \in A$  by the definition of set difference. Also,  $A \cap B \subseteq A$  because each  $x \in A \cap B$  must also have  $x \in A$  by the definition of intersection. Thus,  $(A - B) \cup (A \cap B) \subseteq A$  by the definition of set union and the subset relationship.

To prove (4), note that each  $x \in A - B$  must satisfy  $x \notin B$  by the definition of set difference. Also, each  $x \in A \cap B$  must satisfy  $x \in B$ , by the definition of set intersection. Thus, it is impossible to have  $x \in (A - B) \cap (A \cap B)$ , which proves (4) by the definition of the empty set.

Similar to (1), we also have the following disjoint set union:

$$B = (B - A) \cup (B \cap A) \quad (5)$$

Applying the Sum Principle (on p.1-19) to (1) and (5), we have the following

$$|A| = |A - B| + |A \cap B| \quad (6)$$

and  $|B| = |B - A| + |B \cap A| \quad (7)$

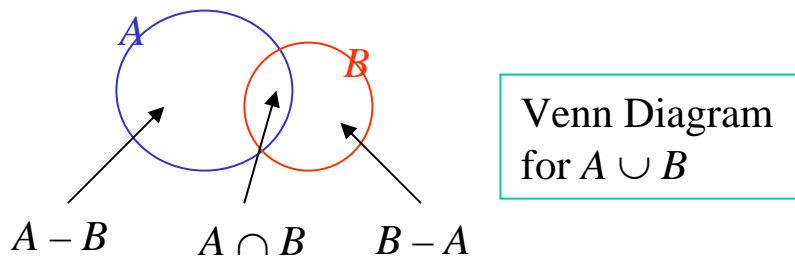
Combining (1) and (5) yields the following disjoint union (the sets  $A - B$  and  $B - A$  are disjoint because  $x \in A - B$  implies  $x \notin B$ , which means  $x \notin B - A$ )

$$A \cup B = (A - B) \cup (B - A) \cup (B \cap A) \quad (8)$$

which implies the following equation using the Sum Principle (p. 1-19):

$$|A \cup B| = |A - B| + |B - A| + |B \cap A| \quad (9)$$

Combining (6), (7), and (9) gives the equation  $|A \cup B| = |A| + |B| - |A \cap B|$ .



**Theorem.** Let  $A$ ,  $B$ , and  $C$  denote three finite sets. Then  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$ .

**Proof:** Applying the previous theorem to sets  $A$  and  $(B \cup C)$ , we have

$$\begin{aligned} |A \cup B \cup C| &= |A \cup (B \cup C)|, \text{ associative law} \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \end{aligned} \quad (1)$$

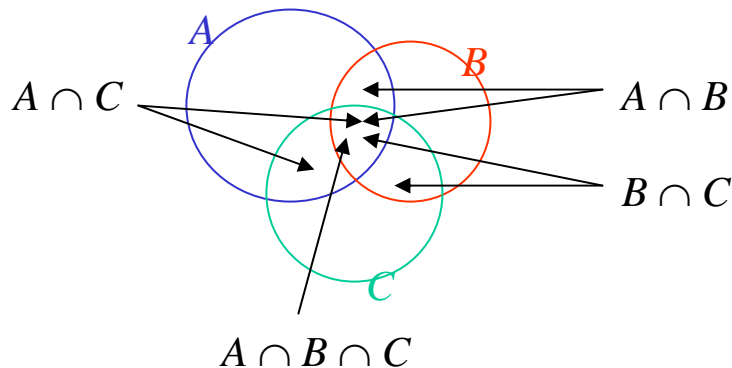
$$\text{Note that } |B \cup C| = |B| + |C| - |B \cap C| \quad (2)$$

And note that  $|A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$ , by distributive law

$$\begin{aligned} &= |A \cap B| + |A \cap C| - |A \cap B \cap A \cap C| \\ &= |A \cap B| + |A \cap C| - |A \cap B \cap C|, \text{ because } A \cap A = A \end{aligned}$$

Thus, substituting the above equation and (2) into (1) proves

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$



Venn Diagram  
for  $A \cup B \cup C$