Problem #1

Let $A$ and $B$ be two arbitrary sets. Let $U$ denote universal set, that is, $A \subseteq U$ and $B \subseteq U$.

Show that $A \subseteq B \iff \overline{B} \subseteq \overline{A}$. Provide all the details of the proof.

Solution:

Thinking flow:

$A \subseteq B$

$\iff \forall x, x \in A \to x \in B$

Definition of subset

$\iff \forall x, \neg(x \in B) \to \neg(x \in B)$

$\iff \forall x, x \notin B \to x \notin A$

Negation of “is element of”

$\iff \forall x, x \notin B \to x \in \overline{A}$

Definition of complement

$\iff \overline{B} \subseteq \overline{A}$

Definition of subset

It remains to prove $(*)$.

Step 1. Let $p$ and $q$ be two propositions. The contraposition rule in propositional logic means that

$$p \to q \iff \neg q \to \neg p$$

Step 2. Let $p$ and $q$ be two unary predicates over $U$

$$\forall x (\neg q(x) \to \neg p(x)) \iff \forall x (p(x) \to q(x))$$

is proved as follows:

(a). $\forall x (p(x) \to q(x)) \iff p(u) \to q(u)$

(b). $p(u) \to q(u)$

Rule of General Specification

(c). $\neg q(u) \to \neg p(u)$

Apply Step 1

(d). $\forall x (\neg q(x) \to \neg p(x))$

Rule of Universal Generalization

Step 3. Set $p(x) : x \in A$ and $q(x) : x \in B$. Apply the contraposition rule in predicate logic established we obtain the desired logical equivalence:

$$\forall x ((x \in A) \to (x \in B)) \iff \forall x (\neg(x \in B) \to \neg(x \in A))$$
Problem #2

Show that $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$. It is sufficient to outline the idea of the proof. (Hint: Refer to the ideas in problem 1.)

Solution:

(a) The logical equivalence $p \rightarrow q \Rightarrow p \land r \rightarrow q \land r$ can be simply verified by checking the two cases $r = 0$ and $r = 1$.

(b) Let $p, q, r$ be arbitrary 1-place (or unary) predicates over $U$. By using the Rule of General Specification, step (a), and then the Rule of Universal Generalization, we obtain the logical implication

$$\forall x (p(x) \rightarrow q(x)) \Rightarrow \forall x (p(x) \land r(x)) \rightarrow (q(x) \land r(x))$$

(c) Set $p(x) := x \in A$, $q(x) := x \in B$, $r(x) := x \in C$. By applying (b) we obtain the following logical implication

$$\forall x (x \in A \land x \in C) \Rightarrow \forall x (x \in B \land x \in C)$$

This logical implication is equivalent to $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$.

Note:
Let $R, S,$ and $T$ be arbitrary sets. From the above, we obtain the following (almost elementary) rules (the third rule is easily proved):

Rule I: $R \subseteq S \Rightarrow \overline{S} \subseteq \overline{R}$

Rule II: $R \subseteq S \Rightarrow (R \cap T) \subseteq (S \cap T)$

Rule III: $R \subseteq S \Rightarrow (R \cup T) \subseteq (S \cup T)$
**Problem #3**

Prove or disprove \( C \subseteq B \Rightarrow [(A - B) \cup (B - C) \subseteq \overline{C} \cap (A \cup B)] \)

We have the following logical implication:

\[
C \subseteq B \\
\Rightarrow \overline{B} \subseteq \overline{C} \quad \text{rule I} \\
\Rightarrow (A \cap \overline{B}) \subseteq (A \cap \overline{C}) \quad \text{rule II} \quad (*) \\
\Rightarrow [(A \cap \overline{B}) \cup (B \cap \overline{C})] \subseteq [(A \cap \overline{C}) \cup (B \cap \overline{C})] \quad \text{rule II} \quad (**)
\]

We have

\[
(A - B) \cup (B - C) \\
= (A \cap \overline{B}) \cup (B \cap \overline{C}) \quad \text{definition of complement} \\
\subseteq (A \cap \overline{C}) \cup (B \cap \overline{C}) \quad \text{because of (*)} \Rightarrow (**) \\
= (A \cup B) \cap \overline{C} \quad \text{distributive law} \\
= \overline{C} \cap (A \cup B) \quad \text{commutative law}
\]

This proves the logical implication \( C \subseteq B \Rightarrow [(A - B) \cup (B - C) \subseteq \overline{C} \cap (A \cup B)] \).
Problem #4

\[ M = \{a, b, c, d\} \] and \( U \) is the power set of \( M \), that is \( U := P(M) \). Let \( isSubset \) be the 2-place (binary) predicate over \( U \) defined by setting

\[ isSubset(A, B) : A \subseteq B \]

Prove or disprove the following statements.

(a) \( \exists A \forall B \ isSubset(A, B) \)

(b) \( \exists A \forall B \ isSubset(B, A) \)

(c) \( \forall A \forall B \ isSubset(A, B) \lor isSubset(B, A) \)

Solution:

(a) True

Simply choose \( A := \emptyset \), then \( \forall B \ isSubset(\emptyset, B) \) is true. Recall that the empty set is subset of every set.

(minimal element of the subset relation; we'll learn that later)

(b) True

Simply choose \( A = M \) then \( \forall B \ isSubset(B, M) \) is true as for i \( B \subseteq M \) for every \( B \) in \( P(M) \).

(maximal element of the subset relation; we'll learn that later).

(c) False

It suffices to find a counter example to disprove the statement. Let \( A = \{a\}, B=\{b\} \) then \( A \not\subseteq B \lor B \not\subseteq A \) is false.

(the subset relation defines only a partial order and not a total order; we'll learn that later).