COT 6600: Quantum Computing

Assignment 2
Out: 09/20/2007 Due: 10/4/2007

1. (5 points) Let A be an arbitrary $m \times m$ matrix. Prove that
$$(A^\dagger)^\dagger = A.$$
To do this, use Theorem 2.3.2 in the textbook to expresss A as a linear combination of outer products.

2. (5 points) Let $W = V_1 \otimes V_2 \cdots \otimes V_n$, where V_i are arbitrary unitary $m \times m$ matrices. Determine the inverse of W.

3. (10 points) Let U be an arbitrary 2×2 unitary matrix and I_2 be the 2×2 identity matrix. Is the 4×4 matrix
$$V := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U + \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2 \quad \text{with } \alpha \in \mathbb{R}$$
unitary? Verify your answer.

4. (15 points) The set of $m \times m$ matrices forms an m^2-dimensional vector space. The trace inner product on this vector space is defined as follows:
$$\langle A, B \rangle := \mathrm{Trace}(A^\dagger B)/m.$$
   (a) (5 points) Consider the matrices
$$E := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } F := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & -\omega^3 \end{pmatrix}$$
   where $\omega = e^{2\pi i/4}$. Calculate $\langle E, F \rangle$.

   (b) (5 points) Prove that $\langle UA, UB \rangle = \langle A, B \rangle$ for all matrices A and B and all unitary matrices U.

   (c) (5 points) Simplify $\langle A \otimes C, B \otimes D \rangle$ where A, B, C, D are arbitrary matrices.

5. (a) (5 points) Find two examples of unitary matrices U with $U^\dagger = U$.

   (b) (5 points) Show that there is only one unitary matrix U with $U^2 = U$.

6. (15 points) Let $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ be the standard ONB in $\mathbb{C}^4$. The cyclic shift operator S permutes the basis vectors as follows: $|0\rangle \to |1\rangle, |1\rangle \to |2\rangle, |2\rangle \to |3\rangle, |3\rangle \to |0\rangle$.

   Let $|\psi\rangle := \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle$. Do the following:

   (a) (9 points) Determine the matrix representation for S^3 and $S^\dagger$

   (b) (3 points) Compute $S^2|\psi\rangle$.

   (c) (3 points) Compute $\langle\psi|(S|\psi\rangle)$.

7. (10 points) Let $U_0, U_1, \ldots, U_{n-1}$ be unitary matrices such that

$$U_0 U_1 \ldots U_{n-1} = I\,.$$

   Show that the product of any cyclic permutation is also equal to the identity matrix, that is:

$$U_k U_{k+1} \ldots U_{n-1} U_0 \ldots U_{k-1} = I$$

   where $k < n$ and the addition of the subscripts is modulo n.

8. (10 points) Do Exercise 3.3.1

9. (15 points) Do Exercise 3.4.1 (a)(b)(c)