COT 6600: Quantum Computing

Assignment 4
Out: 11/26/2007 Due: 12/03/2007

1. (20 points) Do Exercise 8.1.2
2. (20 points) Do Exercise 8.1.3
3. (20 points) Do Exercise 8.1.4
4. (Bonus) Do Exercise 8.1.1
5. (Bonus) Assume an N item search for M solutions, with initial state $|\psi\rangle$. $\sum_x'$ indicates a sum over all x which are solutions to the search problem and $\sum_x''$ indicates a sum over all x which are not solutions to the search problem. Define the normalized states

$$|\alpha\rangle \equiv \frac{1}{\sqrt{N-M}}\sum_x{}''|x\rangle$$

$$|\beta\rangle \equiv \frac{1}{\sqrt{M}}\sum_x{}'|x\rangle$$

Note that the initial state

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_x |x\rangle$$

can be expressed as

$$|\psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle.$$

Let $\cos\theta/2 = \sqrt{N-M/N}$. Now we can rewrite $|\psi\rangle$ as

$$|\psi\rangle = \cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle.$$

   1. Show that in the $|\alpha\rangle$, $|\beta\rangle$ basis, we may write the Grover iteration as

$$G = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

   where θ is a real number in the range 0 to $\pi/2$, chosen so that

$$\sin\theta = \frac{2\sqrt{M(N-M)}}{N}.$$

This means that G is simply a rotation by angle θ in the plane spanned by $|\alpha\rangle$ and $|\beta\rangle$.

The intuition behind Grover's algorithm is that with each application of G we get closer to the state $|\beta\rangle$. Let CI(x) denote the closest integer to the real number x. Repeating the Grover iteration

$$R = \mathrm{CI}(\frac{\arccos\sqrt{\mathrm{M/N}}}{\theta})$$

times rotates $|\psi\rangle$ very close to $|\beta\rangle$. The angle between $G^R|\psi\rangle$ and $|\beta\rangle$ is at most $\theta/2$.

2. Assume $M \ll N$ so we have $\theta \approx \sin\theta \approx 2\sqrt{M/N}$. Show that

$$R \leq \lceil \frac{\pi}{4}\sqrt{\frac{N}{M}} \rceil$$

That is $R = O(\sqrt{N/M})$.