UPW Reflection – Oblique Incidence 2
Equivalent Transmission Line for TE Case (1)

\[
\begin{aligned}
- j k_x E_y &= - j \omega \mu_n H_z \\
- j k_x H_z - \frac{\partial H_x}{\partial z} &= - j \omega \varepsilon_n E_y \\
\frac{\partial E_y}{\partial z} &= j \omega \mu_n H_x
\end{aligned}
\]

(a) \Rightarrow (b): \quad - \frac{\partial H_x}{\partial z} = - j \omega \varepsilon_n E_y + j k_x \frac{k_x}{\omega \mu_n} E_y

= - j \frac{\omega^2 \mu_n \varepsilon_n - k_x^2}{\omega \mu_n} E_y

Since

\[
\begin{aligned}
k_n^2 &= \omega^2 \mu_n \varepsilon_n \\
k_{zn}^2 &= k_n^2 - k_x^2
\end{aligned}
\]

\[
\frac{\partial (-H_x)}{\partial z} = - j \frac{k_{zn}^2}{\omega \mu_n} E_y = - j k_{zn} \frac{k_{zn}}{\omega \mu_n} E_y
\]
\[ \frac{\partial E_y}{\partial z} = -jk_{zn} \frac{\omega \mu_n}{k_{zn}} (-H_x) \]
\[ \frac{\partial (-H_x)}{\partial z} = -jk_{zn} \frac{k_{zn}}{\omega \mu_n} E_y \]

If we let \( V(z) = E_y \), \( I(z) = -H_x \), we can get transmission line equations:

\[ \frac{dV(z)}{dz} = -jk_{zn} Z_{0n} I(z) \]
\[ \frac{dI(z)}{dz} = -jk_{zn} Y_{0n} V(z) \]

with equivalent characteristic impedance for TE case

\[ Z_{0n} = \frac{1}{Y_{0n}} = \frac{\omega \mu_n}{k_{zn}} \]
Equivalent Transmission Line for TM Case (1)

\[
\begin{aligned}
- j k_x H_y &= j \omega \varepsilon_n E_z \\
- j k_x E_z - \frac{\partial E_x}{\partial z} &= j \omega \mu_n H_y \\
\frac{\partial H_y}{\partial z} &= - j \omega \varepsilon_n E_x
\end{aligned}
\]

(a) \Rightarrow (b): \quad \frac{\partial E_x}{\partial z} = - j \omega \varepsilon_n H_y + j k_x \frac{k_x}{\omega \varepsilon_n} H_y

\[
= - j \frac{\omega^2 \mu_n \varepsilon_n - k_x^2}{\omega \varepsilon_n} H_y
\]

Since \[ k_n^2 = \omega^2 \mu_n \varepsilon_n \]
\[ k_{zn}^2 = k_n^2 - k_x^2 \quad \Rightarrow \quad \frac{\partial E_x}{\partial z} = - j \frac{k_{zn}^2}{\omega \varepsilon_n} H_y = - j k_{zn} \frac{k_{zn}}{\omega \varepsilon_n} H_y \]
Equivalent Transmission Line for TM Case (2)

\[ \begin{align*}
\frac{\partial E_x}{\partial z} &= -j k_{zn} \frac{k_{zn}}{\omega \varepsilon_n} H_y \\
\frac{\partial H_y}{\partial z} &= -j k_{zn} \frac{\omega \varepsilon_n}{k_{zn}} E_x
\end{align*} \]

If we let \( V(z) = E_x \), \( I(z) = H_y \), we can get transmission line equations:

\[ \begin{align*}
\frac{dV(z)}{dz} &= -j k_{zn} Z_{0n} I(z) \\
\frac{dI(z)}{dz} &= -j k_{zn} Y_{0n} V(z)
\end{align*} \]

with equivalent characteristic impedance for TM case

\[ Z_{0n} = \frac{1}{Y_{0n}} = \frac{k_{zn}}{\omega \varepsilon_n} \]
Important Notice: In z direction, we use $k_{zn}$ instead of $k_n$.

$$V_n(z) = Ae^{-jk_{zn}z} + Be^{jk_{zn}z}$$

$$I_n(z) = \frac{A}{Z_{0n}} e^{-jk_{zn}z} - \frac{B}{Z_{0n}} e^{jk_{zn}z}$$

$Z_{0n} = \begin{cases} \frac{\omega \mu_n}{k_{zn}} & \text{for TE case} \\ \frac{k_{zn}}{\omega \varepsilon_n} & \text{for TM case} \end{cases}$

Define: guide wavelength for nth region: $\lambda_{gn} = \frac{2\pi}{k_{zn}}$
Fresnel’s Law Derived from Transmission Line Analysis (1)
Fresnel’s Law Derived from Transmission Line Analysis (2)

\[
\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \quad Z_{0n} = \begin{cases} 
\frac{\omega \mu_n}{k_{zn}} & \text{for TE case} \\
\frac{k_{zn}}{\omega \varepsilon_n} & \text{for TM case} 
\end{cases} \quad n = 1,2
\]

\[
k_{z1} = k_1 \cos \theta_i, \quad k_{z2} = k_2 \cos \theta_t, \quad k_n = \omega \sqrt{\mu_n \varepsilon_n}, \quad \eta_n = \sqrt{\frac{\mu_n}{\varepsilon_n}}
\]

\[
Z_{0n} = \begin{cases} 
\frac{\eta_n}{\cos \theta_n} & \text{for TE (or \perp) case} \\
\eta_n \cos \theta_n & \text{for TM (or \parallel) case} 
\end{cases}
\]

\[
\Gamma_\perp = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \quad \Gamma_\parallel = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}
\]
Total Internal Reflection (1)

For lossless nonmagnetic media, happens when \( n_1 > n_2 \).

From the law of refraction:

\[
\frac{\sin \theta_i}{\sin \theta_t} = \frac{n_2}{n_1}
\]

When \( \theta_t = 90^\circ \), critical incident angle \( \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \)

When \( \theta_i \geq \theta_c \), total internal reflection will happen.

This is true for both TE and TM cases.
Total Internal Reflection (2)

Find \( |\Gamma_\perp| \) and \( |\Gamma_\parallel| \) for total internal reflection.

\[
\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}
\]

\[
\theta_i > \theta_c, \quad \Rightarrow \sin \theta_i > \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}
\]

\[
\Rightarrow k_x = k_1 \sin \theta_i > k_0 n_1 \frac{n_2}{n_1} = k_0 n_2 = k_2
\]

\[
k_{z2}^2 = k_2^2 - k_x^2 < 0 \quad \Rightarrow \quad k_{z2} \text{ becomes purely imaginary, but}
\]

\[
k_{z1} = k_1 \cos \theta_i \text{ still real}
\]

\[
Z_{02\perp} = \frac{\omega \mu_0}{k_{z2}} \text{ Purely imaginary}
\]

\[
Z_{02\parallel} = \frac{k_{z2}}{\omega \varepsilon_2} \text{ Purely imaginary}
\]

\[
\Gamma_{\perp or \parallel} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \Rightarrow \quad |\Gamma_{\perp or \parallel}| = 1
\]
Total Internal Reflection (3)

Note in region 2

\[ e^{-jk_2z} e^{-jk_x x} = e^{-j(\gamma_2 z)} e^{-jk_x x} \]

\[ = e^{-\gamma_2 z} e^{-jk_x x} \]

Constant phase plane \( x = \text{constant} \).

Constant amplitude plane \( z = \text{constant} \).

Non-uniform plane wave
Example 5-6. Determine the range of values of the dielectric constant of a dielectric slab of thickness \( t \) so that, when a wave is incident on it from one of its ends at an oblique angle \( 0^\circ \leq \theta_i \leq 90^\circ \), the energy of the wave in the dielectric is contained within the slab. The geometry of the problem is shown in the Figure 5-8.

Solution. We assume that the slab width is infinite. To contain the energy of the wave within the slab, the reflection angle \( \theta_r \) of the wave bouncing within the slab must be equal to or greater than the critical angle \( \theta_c \). By referring to Figure 5-8, the critical angle can be related to the refraction angle \( \theta_i \) by

\[
\sin \theta_r = \sin \left( \frac{\pi}{2} - \theta_i \right) = \cos \theta_i \geq \sin \theta_c = \sqrt{\frac{\varepsilon_0}{\varepsilon_r \varepsilon_0}} = \frac{1}{\sqrt{\varepsilon_r}}
\]

or

\[
\cos \theta_i \geq \frac{1}{\sqrt{\varepsilon_r}}
\]

FIGURE 5-8 Dielectric slab of thickness \( t \) and wave containment within.
At the interface formed at the leading edge, Snell’s law of refraction must be satisfied. That is,

\[ \beta_0 \sin \theta_i = \beta_1 \sin \theta_i \Rightarrow \sin \theta_i = \frac{\beta_0}{\beta_1} \sin \theta_i = \frac{1}{\sqrt{\varepsilon_r}} \sin \theta_i \]

Using this, we can write the aforementioned \( \cos \theta_i \) as

\[ \cos \theta_i = \sqrt{1 - \sin^2 \theta_i} = \sqrt{1 - \frac{1}{\varepsilon_r} \sin^2 \theta_i} \geq \frac{1}{\sqrt{\varepsilon_r}} \]

or

\[ \sqrt{1 - \frac{1}{\varepsilon_r} \sin^2 \theta_i} \geq \frac{1}{\sqrt{\varepsilon_r}} \]

Solving this leads to

\[ \varepsilon_r - \sin^2 \theta_i \geq 1 \]

or

\[ \varepsilon_r \geq 1 + \sin^2 \theta_i \]

To accommodate all the angles, the dielectric constant must be

\[ \varepsilon_r \geq 2 \]

since the smallest and largest values of \( \theta_i \) are, respectively, \( 0^\circ \) and \( 90^\circ \). This is achievable by many practical dielectric material such as Teflon (\( \varepsilon_r \approx 2.1 \)), polystyrene (\( \varepsilon_r \approx 2.56 \)), and many others.
A. PERPENDICULAR POLARIZATION

To see the conditions under which the reflection coefficient of (5-17a) will vanish, we set it equal to zero, which leads to

$$
\Gamma_\perp = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}} \cos \theta_i - \sqrt{\frac{\mu_1}{\varepsilon_1}} \cos \theta_t}{\sqrt{\frac{\mu_2}{\varepsilon_2}} \cos \theta_i + \sqrt{\frac{\mu_1}{\varepsilon_1}} \cos \theta_t} = 0
$$

(5-27)

or

$$
\cos \theta_i = \sqrt{\frac{\mu_1}{\mu_2} \left( \frac{\varepsilon_2}{\varepsilon_1} \right)} \cos \theta_t
$$

(5-27a)

Using Snell’s law of refraction, as given by (5-15b), (5-27a) can be written as

$$
(1 - \sin^2 \theta_i) = \frac{\mu_1}{\mu_2} \left( \frac{\varepsilon_2}{\varepsilon_1} \right)(1 - \sin^2 \theta_t)
$$

$$
(1 - \sin^2 \theta_i) = \frac{\mu_1}{\mu_2} \left( \frac{\varepsilon_2}{\varepsilon_1} \right) \left[ 1 - \frac{\mu_1}{\mu_2} \left( \frac{\varepsilon_1}{\varepsilon_2} \right) \sin^2 \theta_i \right]
$$

(5-28)
or

\[
\sin \theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}} \quad (5-28a)
\]

Since the sine function cannot exceed unity, (5-28a) exists only if

\[
\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1} \leq \frac{\mu_1}{\mu_2} - \frac{\mu_2}{\mu_1} \quad (5-29)
\]

or

\[
\frac{\varepsilon_2}{\varepsilon_1} \leq \frac{\mu_1}{\mu_2} \quad (5-29a)
\]

If however \( \mu_1 = \mu_2 \), (5-28a) indicates that

\[
\sin \theta_i |_{\mu_1=\mu_2} = \infty \quad (5-29b)
\]

Therefore there exists no real angle \( \theta_i \) under this condition that will reduce the reflection coefficient to zero. Since the permeability for most dielectric material (aside from ferromagnetics) is almost the same and equal to that of free space \( (\mu_1 = \mu_2 = \mu_0) \), then for these materials there exists no real incidence angle that will reduce the reflection coefficient for perpendicular polarization to zero.
B. PARALLEL POLARIZATION

To examine the conditions under which the reflection coefficient for parallel polarization will vanish, we set (5-24c) equal to zero; that is

\[
\Gamma''_n = \frac{-\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos \theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos \theta_t}{\sqrt{\frac{\mu_1}{\varepsilon_1}} \cos \theta_i + \sqrt{\frac{\mu_2}{\varepsilon_2}} \cos \theta_t} = 0
\]  

(5-30)

or

\[
\cos \theta_i = \sqrt{\frac{\mu_2}{\mu_1} \left(\frac{\varepsilon_1}{\varepsilon_2}\right)} \cos \theta_t
\]  

(5-30a)

Using Snell's law of refraction, as given by (5-24b), (5-30a) can be written as

\[
(1 - \sin^2 \theta_i) = \frac{\mu_2}{\mu_1} \left(\frac{\varepsilon_1}{\varepsilon_2}\right) (1 - \sin^2 \theta_t)
\]

\[
(1 - \sin^2 \theta_i) = \frac{\mu_2}{\mu_1} \left(\frac{\varepsilon_1}{\varepsilon_2}\right) \left(1 - \frac{\mu_1}{\mu_2} \left(\frac{\varepsilon_1}{\varepsilon_2}\right) \sin^2 \theta_t\right)
\]

(5-31)

or

\[
\sin \theta_i = \sqrt{\frac{\varepsilon_2 - \frac{\mu_2}{\varepsilon_1}}{\frac{\mu_1}{\varepsilon_1} \frac{\mu_1}{\varepsilon_2} - \frac{\varepsilon_1}{\varepsilon_2}}} \quad \text{(5-31a)}
\]
Total Transmission (4)

Since the sine function cannot exceed unity, (5-31a) exists only if

\[
\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1} \leq \frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}
\]

(5-32)

or

\[
\frac{\mu_2}{\mu_1} \geq \frac{\varepsilon_1}{\varepsilon_2}
\]

(5-32a)

If however \( \mu_1 = \mu_2 \), (5-31a) reduces to

\[
\theta_i = \theta_B = \sin^{-1} \left( \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}} \right)
\]

(5-33)

The incidence angle \( \theta_i \), as given by (5-31a) or (5-33), which reduces the reflection coefficient for parallel polarization to zero, is referred to as the Brewster angle \( \theta_B \). It should be noted that when \( \mu_1 = \mu_2 \), the incidence Brewster angle \( \theta_i = \theta_B \) of (5-33) exists only if the polarization of the wave is parallel.

Other forms of the Brewster angle, besides that given by (5-33), are

\[
\theta_i = \theta_B = \cos^{-1} \left( \sqrt{\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}} \right)
\]

(5-33a)

\[
\theta_i = \theta_B = \tan^{-1} \left( \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \right)
\]

(5-33b)
Total Transmission (5)

TE incidence $\mu_1 = \mu_2 = \mu_0$

TM incidence
Polarization Characteristics (1)

Incident wave: $\mathbf{E}_{i//}, \mathbf{E}_{i\perp}$, and $\mathbf{k}_i$ form a right handed system
Reflected wave: $\mathbf{E}_{r\perp}, \mathbf{E}_{r//},$ and $\mathbf{k}_r$ form a right handed system
Transmitted wave: $\mathbf{E}_{t//}, \mathbf{E}_{t\perp},$ and $\mathbf{k}_t$ form a right handed system
Polarization Characteristics (2)

Incident wave: $\mathbf{E}_{i\parallel}$, $\mathbf{E}_{i\perp}$, and $\mathbf{k}_i$ form a right handed system
Reflected wave: $\mathbf{E}_{r\perp}$, $\mathbf{E}_{r\parallel}$, and $\mathbf{k}_r$ form a right handed system
Transmitted wave: $\mathbf{E}_{t\parallel}$, $\mathbf{E}_{t\perp}$, and $\mathbf{k}_t$ form a right handed system

$$\Gamma_{\parallel} = \frac{E_{0r\parallel}}{E_{0i\parallel}} \quad T_{\parallel} = \frac{E_{0t\parallel}}{E_{0i\parallel}} \quad \Gamma_{\perp} = \frac{E_{0r\perp}}{E_{0i\perp}} \quad T_{\perp} = \frac{E_{0r\perp}}{E_{0i\perp}}$$

From the polarization of incident wave and Fresnel’s formula, we can find the polarization of reflected and transmitted wave.

For normal incidence, since $\Gamma_{\parallel} = \Gamma_{\perp}$ and $T_{\parallel} = T_{\perp}$:

if the incident is RHP, the reflected must be LHP and transmitted must still be RHP;
if the incident is LHP, the reflected must be RHP and transmitted must still be LHP.
Dielectric-Conductor Interface (1)

Let us assume that a uniform plane wave is obliquely incident upon a planar interface where medium 1 is a perfect dielectric and medium 2 is lossy, as shown in Figure 5-10 [3]. For either the perpendicular or parallel polarization, the transmitted electric field into medium 2 can be written, using modified forms of either (5-12a) or (5-22a), as

$$E' = E_2 \exp \left[ -\gamma_2 \left( x \sin \theta_t + z \cos \theta_t \right) \right] = E_2 \exp \left[ -\left( \alpha_2 + j\beta_2 \right) \left( x \sin \theta_t + z \cos \theta_t \right) \right]$$

(5-53)

It can be shown that for lossy media Snell’s law of refraction can be written as

$$\gamma_1 \sin \theta_i = \gamma_2 \sin \theta_t$$

(5-54)

Therefore for the geometry of Figure 5-10,

$$\sin \theta_t = \frac{\gamma_1}{\gamma_2} \sin \theta_i = \frac{j\beta_1}{\alpha_2 + j\beta_2} \sin \theta_i$$

(5-55a)
and
\[
\cos \theta_i = \sqrt{1 - \sin^2 \theta_i} = \sqrt{1 - \left( \frac{j \beta_1}{\alpha_2 + j \beta_2} \right)^2} = \sin^2 \theta_i = se^{i\xi} = s(\cos \xi + j \sin \xi)
\]

Using (5-55a) and (5-55b) we can write (5-53) as
\[
E' = E_2 \exp \left( - (\alpha_2 + j \beta_2) \left( x \frac{j \beta_1}{\alpha_2 + j \beta_2} \sin \theta_i + zs(\cos \xi + j \sin \xi) \right) \right)
\]
which reduces to
\[
E' = E_2 \exp \left[ -zs(\alpha_2 \cos \xi - \beta_2 \sin \xi) \right] \\
\times \exp \left\{ -j \left[ \beta_1 x \sin \theta_i + zs(\alpha_2 \sin \xi + \beta_2 \cos \xi) \right] \right\}
E' = E_2 e^{-zp} \exp \left[ -j(\beta_1 x \sin \theta_i + zq) \right]
\]
where
\[
p = s(\alpha_2 \cos \xi - \beta_2 \sin \xi) = \alpha_2 e \\
q = s(\alpha_2 \sin \xi + \beta_2 \cos \xi)
\]
It is apparent that (5-57) represents a nonuniform wave.
Dielectric-Conductor Interface (3)
Dielectric-Conductor Interface (4)

The instantaneous field of (5-57) can be written, assuming $E_2$ is real, as

$$\mathcal{E}' = \text{Re}(E'e^{j\omega t}) = E_2 e^{-zp} \text{Re}\left\{ j \left[ \omega t - (\beta_1 x \sin \theta_i + zq) \right] \right\}$$

$$\mathcal{E}' = E_2 e^{-zp} \cos \left[ \omega t - (\beta_1 x \sin \theta_i + zq) \right]$$  \hspace{1cm} (5-58)

The constant amplitude planes ($z = \text{constant}$) of (5-58) are parallel to the interface, and they are shown dashed-dotted in Figure 5-10. The constant phase planes $[\omega t - (kx \sin \theta_i + zq) = \text{constant}]$ are inclined at an angle $\psi_2$ that is no longer $\theta_i$.

To determine the constant phase we write the argument of the exponential or of the cosine function in (5-58) as

$$\omega t - (\beta_1 x \sin \theta_i + zq) = \omega t - \sqrt{(\beta_1 x \sin \theta_i)^2 + q^2}$$

$$\times \left[ \frac{(\beta_1 x \sin \theta_i)}{\sqrt{(\beta_1 x \sin \theta_i)^2 + q^2}} + \frac{q z}{\sqrt{(\beta_1 x \sin \theta_i)^2 + q^2}} \right]$$  \hspace{1cm} (5-59)

If we define an angle $\psi_2$ such that

$$u = \beta_1 \sin \theta_i$$  \hspace{1cm} (5-60a)

$$\sin \psi_2 = \frac{\beta_1 \sin \theta_i}{\sqrt{(\beta_1 \sin \theta_i)^2 + q^2}} = \frac{u}{\sqrt{u^2 + q^2}}$$  \hspace{1cm} (5-60b)

$$\cos \psi_2 = \frac{q}{\sqrt{(\beta_1 \sin \theta_i)^2 + q^2}} = \frac{q}{\sqrt{u^2 + q^2}}$$  \hspace{1cm} (5-60c)
or

\[
\psi_2 = \tan^{-1}\left(\frac{\beta_1 \sin \theta_i}{q}\right) = \tan^{-1}\left(\frac{u}{q}\right) \tag{5-60d}
\]

we can write (5-59) and in turn (5-58) as

\[
\delta'^t = E_2 e^{-z \rho} \text{Re}\left(\exp\left\{ j \left[ \omega t - \sqrt{u^2 + q^2} \left( \frac{ux}{\sqrt{u^2 + q^2}} + \frac{qz}{\sqrt{u^2 + q^2}} \right) \right]\right\}\right)
\]

\[
= E_2 e^{-z \rho} \text{Re}\left(\exp\left\{ j \left[ \omega t - \beta_{2e} \left( x \sin \psi_2 + z \cos \psi_2 \right) \right]\right\}\right)
\]

\[
\delta'^t = E_2 e^{-z \rho} \text{Re}\left(\exp\left\{ j \left[ \omega t - \beta_{2e} (\hat{n}_\psi \cdot \mathbf{r}) \right]\right\}\right) \tag{5-61}
\]

where

\[
\hat{n}_\psi = \hat{\alpha}_x \sin \psi_2 + \hat{\alpha}_z \cos \psi_2 \tag{5-61a}
\]

\[
\beta_{2e} = \sqrt{u^2 + q^2} \tag{5-61b}
\]

It is apparent from (5-60a) through (5-61a) that

1. The true angle of refraction is \(\psi_2\) and not \(\theta_i\) (\(\theta_i\) is complex).
2. The wave travels along a direction defined by unit vector \(\hat{n}_\psi\).
3. The constant phase planes are perpendicular to unit vector \(\hat{n}_\psi\), and they are shown as dashed lines in Figure 5-10.
Multilayer Oblique Incidence Analysis (1)

\[ \mathbf{A}_{\text{tot}} = \mathbf{A}_{N-1} \mathbf{A}_{N-2} \cdots \mathbf{A}_3 \mathbf{A}_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \]

\[ \mathbf{A}_n = \begin{bmatrix} \cos k_{zn} d_n & -jZ_{0n} \sin k_{zn} d_n \\ -j \frac{\sin k_{zn} d_n}{Z_{0n}} & \cos k_{zn} d_n \end{bmatrix} \quad (n = 2, 3, \ldots, N - 1) \]
Multilayer Oblique Incidence Analysis (2)

Define:
\[
\Gamma = \frac{V^-_{01}}{V^+_{01}} \quad \text{and} \quad T = \frac{V^+_{LN}}{V^+_{01}}
\]

for equivalent transmission line

\[
\begin{bmatrix}
T \\
T / Z_{0N}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
1 + \Gamma \\
(1 - \Gamma) / Z_{01}
\end{bmatrix}
\]

Can solve \( T \) and \( \Gamma \) using MatLab
For TE case

\[
E_{0i\perp} = E_{yi} = V_{01}^+, \quad E_{0r\perp} = E_{yr} = V_{01}^-, \quad E_{Lt\perp} = E_{yt} = V_{LN}^+
\]

\[\Rightarrow \Gamma_{\perp} = \Gamma, \quad T_{\perp} = T\]

For TM case

\[
E_{0i\parallel} \cos \theta_i = E_{xi\parallel} = V_{01}^+
\]

\[
E_{0r\parallel} \cos \theta_i = E_{xr\parallel} = V_{01}^-
\]

\[
E_{Lt\parallel} \cos \theta_t = E_{xt\parallel} = V_{LN}^+
\]

\[\Rightarrow \Gamma_{\parallel} = \frac{E_{0r\parallel}}{E_{0i\parallel}} = \Gamma, \quad T_{\parallel} = \frac{E_{Lt\parallel}}{E_{0i\parallel}} = T \frac{\cos \theta_i}{\cos \theta_t}\]
For TE case: Average pointing vector of incident wave

\[ S_i = \frac{1}{2} \text{Re}[E_i \times H_i^*] \]

\[ = \frac{1}{2} \text{Re}(E_{yi} H_{xi}^*)(-a_z) + \frac{1}{2} \text{Re}(E_{yi} H_{zi}^*)a_x \]

For per unit area on xoy plane, power incident into the system

\[ P_i = S_i \cdot a_z = \frac{1}{2} \text{Re}[E_{yi}(-H_{xi}^*)] \]

Using transmission line analogy: \( E_{yi} = V_{01}^+, \quad H_{xi}^* = -I_{01}^+ \)

\[ P_i = \frac{1}{2} \text{Re}[V_{01}^+ I_{01}^{+*}] = \frac{|V_{01}^+|^2}{2} \text{Re}\left( \frac{1}{Z_{01}} \right) \]

Likewise for TM case: \( P_i = \frac{1}{2} \text{Re}[E_{xi} H_{yi}^*] = \frac{1}{2} \text{Re}[V_{01}^+ I_{01}^{+*}] = \frac{|V_{01}^+|^2}{2} \text{Re}\left( \frac{1}{Z_{01}} \right) \)
Power of Incident, Reflected and Transmitted Waves (2)

Similarly, for both TE and TM cases:

\[ P_r = \frac{1}{2} \text{Re}[V_{01}^- I_{01}^-] = \frac{|V_{01}^-|^2}{2} \text{Re}\left(\frac{1}{Z_{01}}\right) \]

\[ P_t = \frac{1}{2} \text{Re}[V_{LN}^+ I_{LN}^+] = \frac{|V_{LN}^+|^2}{2} \text{Re}\left(\frac{1}{Z_{0N}}\right) \]

Define: Reflectance \( R = \frac{P_r}{P_i} = |\Gamma|^2 \)

Transmittance \( T \) is:

\[ T = \frac{P_t}{P_i} = |T|^2 \frac{\text{Re}\left(\frac{1}{Z_{0N}}\right)}{\text{Re}\left(\frac{1}{Z_{01}}\right)} = |T|^2 \frac{Z_{01}}{Z_{0N}} \]

For lossless system \( P_r + P_t = P_i \)

or Reflectance + Transmittance = 1

Regions 1 and N are lossless.