Note 9/11/2012

① $\underbrace{0\,0\,0\,1}_{Y_1}\ \underbrace{0\,0\,0\,0\,1}_{Y_2}\ \underbrace{0\,0\,1}_{Y_3}\ \underbrace{0\,1}_{Y_4}$

Y_i: # of trials until head in cycle i

$Y_i \sim$ Geometric distr.

② $\underbrace{0\,0\,0\,1}_{Y_1}\ \underbrace{0\,0\,0\,0\,1}_{Y_2}\ \underbrace{0\,0\,1}_{Y_3}\ \underbrace{1}_{Y_4}$

r.v. $N \to [2, \infty)$

$Y_1 \sim Y_n$

$$p(N=n) = p(Y_1 \geq 3,\ Y_2 \geq 3,\ \cdots\ Y_{n-1} \geq 3,\ Y_n \leq 2)$$

$$= p(Y_1 \geq 3) \cdot p(Y_2 \geq 3) \cdots p(Y_n \leq 2)$$

$$p(Y_i \leq 2) = p(Y_i = 1) + p(Y_i = 2) = p + (1-p)\cdot p = 2p - p^2$$

$$p(Y_i \geq 3) = 1 - p(Y_i \leq 2) = 1 - 2p + p^2 = (1-p)^2$$

$$\therefore\ p(N=n) = (1-p)^{2n-2} \cdot (2p - p^2).$$

$$\therefore\ E[N] = \sum_{n=2}^{\infty} p(N=n)\cdot n = p(2-p)\cdot \sum_{n=2}^{\infty} n(1-p)^{2n-2}$$

$$E[N] = p(2-p) \cdot \sum_{n=2}^{\infty} n(1-p)^{2(n-1)}$$

define $\alpha = (1-p)^2$ $\hookrightarrow$ $\displaystyle\sum_{n=2}^{\infty} n \cdot \alpha^{n-1} = S$

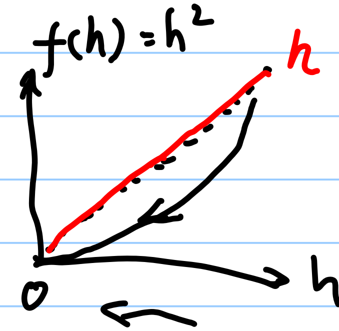
$$S = 2\alpha + 3\alpha^2 + 4\alpha^3 + 5\alpha^4 + \cdots$$

$$-)\quad \alpha S = 2\alpha^2 + 3\alpha^3 + 4\alpha^4 + \cdots$$

$$(1-\alpha)S = \alpha + \underbrace{\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots}_{\to \frac{\alpha}{1-\alpha}}$$

$$= \frac{2\alpha - \alpha^2}{1-\alpha}$$

$$\Rightarrow S = \frac{2\alpha - \alpha^2}{(1-\alpha)^2}$$



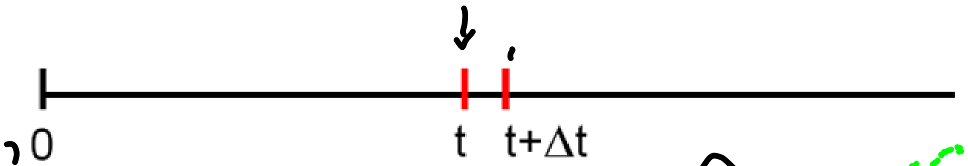
? why?

$$
\begin{aligned}
P[X \leq t + h | X > t] &= P[X \leq h] \\
&= 1 - e^{-\lambda h} \\
&= 1 - [1 - \lambda h + \underbrace{\sum_{n=2}^{\infty} \frac{(\lambda h)^n}{n!}}_{o(h)}] \\
&= \lambda h + o(h)
\end{aligned}
$$

if h is small

$$P(N(h)=0) = 1 - P(N(h)=1) - P(N(h)\geq 2)$$

$$= 1 - \lambda h - o(h) - o(h) = 1 - \lambda h + o(h)$$



▫ P(N(h)=0) = 1 -λh +o(h)

meaning?

① ②

$$
\begin{aligned}
P_n(t+\Delta t) &= P_{n-1}(t)\lambda\Delta t + P_n(t)(1-\lambda\Delta t) + o(\Delta t) \\
P_n(t+\Delta t) - P_n(t) &= P_{n-1}(t)\lambda\Delta t - P_n(t)\lambda\Delta t + o(\Delta t) \\
\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} &= P_{n-1}(t)\lambda - P_n(t)\lambda + \frac{o(\Delta t)}{\Delta t}
\end{aligned}
$$

① n-1 arrived by t, 1 arrived during [t, t+Δt]

② n arrived by t, 0 arrived during [t, t+Δt]

- $dP_0(t)/dt = -\lambda\, P_0(t)$

$P_0(t) = c\, e^{-\lambda t}$ we know $P_0(0) = 1$

$\Rightarrow c = 1$